MSc in Economics for Development

Macroeconomics for Development

Week 3 Class

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Consultation hours: Friday, 2-3pm, Weeks 1,3-8 (MT)

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Week 2 Review

1. Direction of rotation with complex eigenvalues:

Eigenvalue	Stability	Direction of rotation
$a \pm bi$	Stable oscillations: a<0	Test using $\dot{x} = Ax$
	Unstable oscillations: a>0	
	Repeating oscillations: a=0	Will depend on signs of both a and b

2. Correction: Slide 11 and 12. Signs of a,b,c,d should reflect those on slide 10.

1. Do we need “distinct” or “non-zero” eigenvalues to have linearly independent eigenvectors (Math notes, pp105-106)?
   – Eigenvectors from distinct eigenvalues are linearly independent
      • See footnote, pp585, Barro and Sala-i-Martin
   – Eigenvectors from non-distinct eigenvalues may or may-not be linearly independent, see board.

References

- Chung, J. W., 1994, *Utility and Production Functions*, Blackwell, Oxford UK
  - Introduction to key utility functions and their properties
- Varian, H., 1992, *Microeconomic Analysis*, W. W. Norton & Co, UK: Ch 7-9
  - Comprehensive look at utility and demand
- Varian, H., 2003, *Intermediate Microeconomics*, W. W. Norton & Co, UK Ch 4-6
  - Introductory look at utility and demand, but good use of graphs

Week 3 Overview: Utility functions

1. A utility function is a functional representation of consumer preferences
2. To make utility functions easy to use we often also assume some extra characteristics: monotonicity, local non-satiation and convexity
3. We use utility functions to derive demand curve, by choosing the mix of goods that maximises utility subject to a budget constraint
4. The Engel curve then describes how the demand changes with income
5. We will look in depth at three utility functions:
    1. The Cobb-Douglas utility function
    2. The Stone-Geary utility function
    3. The Constant Elasticity of Substitution utility function

A utility function is a functional representation of consumer preferences

A utility function is a functional representation of consumer preferences

Delorean's Theorem (1954): To be represented in a utility function, preferences must be:

1. Complete

For all *x* and *y* in X:

$$x \succeq y \text{ or } y \succeq x \text{ or both}$$

- Preferences over every possibility

2. Reflexive

For all *x in* X:

$$x \succeq x$$

- Each good is at least as good as itself (logical)

3. Transitive

For all *x, y* and *z* in X:

$$\text{if } x \succeq y \text{ and } y \succeq z, \text{ then } x \succeq z$$

- Preferences are consistent and not cyclical

4. Continuous

For all *x* and *y in* X:

$$\text{The sets } \mathbf{x} : \mathbf{x} \succeq \mathbf{y} \text{ and } \mathbf{x} : \mathbf{x} \preceq \mathbf{y} \text{ are closed}$$

- This means all preferences are continuous

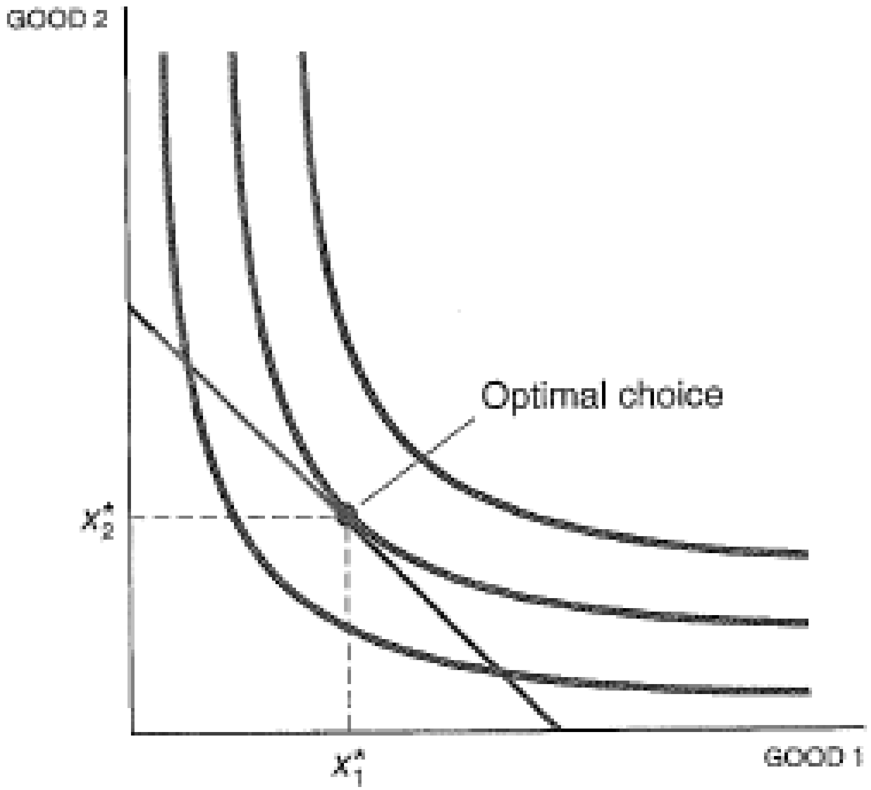
Source: Varian, Microeconomic Analysis, Ch 7.

To make utility functions easy to use we often also assume some extra characteristics: monotonicity, local non-satiation and convex preferences

Monotonicity	**Weak:** • More goods are at least as good as less, as you can throw away items of negative or zero value **Strong:** • More goods are strictly as good as less as no goods have negative or zero value.
Local Non-Satiation	• For any consumption set in the preference space, you must be able to find a preferred consumption bundle by moving in some direction: • No bliss points • No thick indifference curves
Convex preferences	• Agents prefer averages to extremes •Note relationship with quasi-concave utility functions – see Math crash course notes

Source: Chung, Ch 1.,;Varian, Microeconomic Analysis, Ch. 7

To derive a demand curve we chose the mix of goods that maximises utility subject to a budget constraint



The Dual Problem

max $u(\mathbf{x})$ (1)
Such that $\mathbf{px} \leq m$

min $\mathbf{px}$ (2)
Such that $u(\mathbf{x}) \geq u$

Assume that:

- $u(\mathbf{x})$ is continuous
- Preferences satisfy local non-satiation
- Answers to both problems exist

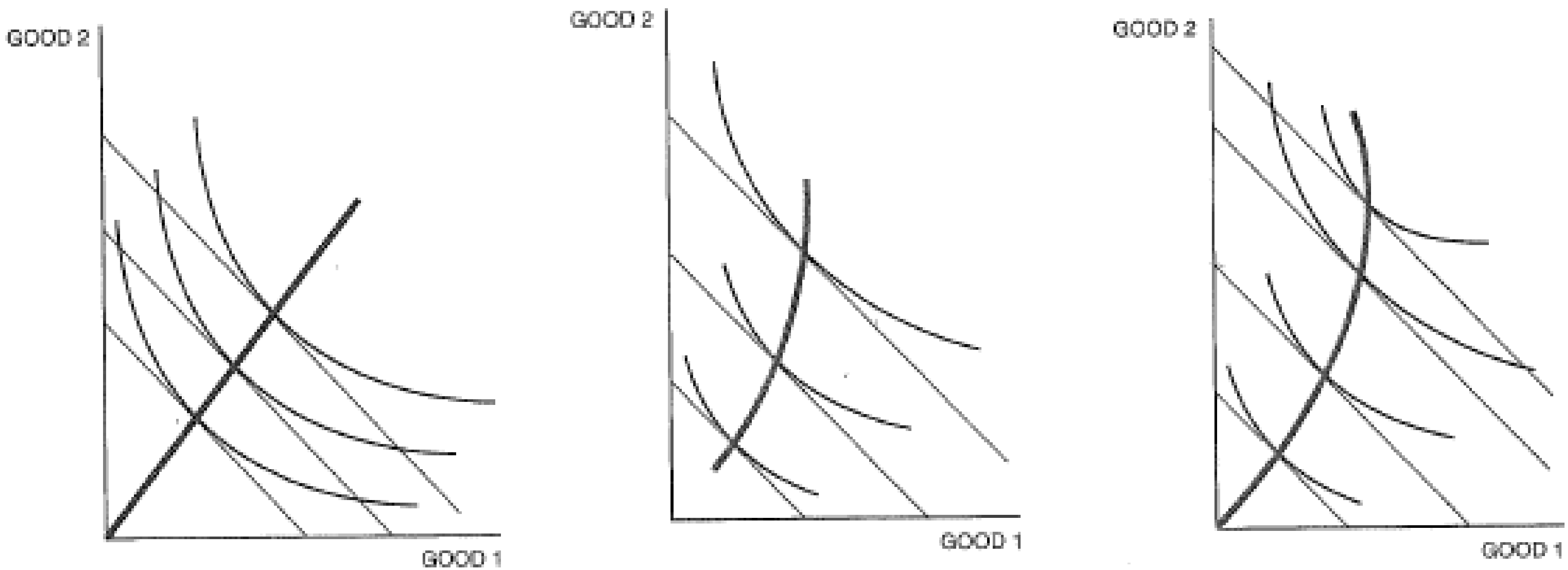
(1) implies (2) and (2) implies (1). Proof: see Varian Ch7 appendix.

A consumption bundle that maximises utility subject to an expenditure constraint will also minimise expenditure subject to a budget constraint and vice versa, under certain assumptions.

Source: Varian, Microeconomic Analysis, Ch 7.

Engel Curves are a representation of how demand changes with income

Unit elastic demands | **Luxury good** | **Inferior good**



Source: Varian, Microeconomic Analysis, Ch 7.

1. The Cobb-Douglas Utility Function:
a. The function and its basic properties

	Formula	Intuition
Utility Function	$U = \prod_{i=1}^{n} q_i^{\beta_i}$ $u = \ln U = \sum_{i=1}^{n} \beta_i \ln q_i$ $0 < \beta_i < 1$	• Multiplicative so must consume a little of everything • Preferences incorporated using β • *U* is utility, *u* is a transformation.
Properties	Monotonic: $\frac{\partial u}{\partial q_i} = \frac{\beta_i}{q_i} > 0$	• Marginal utility of each good is positive
	Concave utility function $\frac{\partial^2 u}{\partial q_i^2} = -\frac{\beta_i}{q_i^2} < 0$	• Marginal utility of each good is decreasing
	Strongly additive $\frac{\partial^2 u}{\partial q_i \partial q_j} = 0$	• Marginal utility of good *i* is independent of good *j*.
	Homothetic $u + \ln\theta \sum_i \beta_i = \sum_i \beta_i \ln(\theta q_i)$	• Utility rises by a scalar proportion if each commodity is multiplied by a scalar

1. The Cobb-Douglas Utility Function:
b. Deriving consumer demand

Utility Function and Budget Constraint

$$u = \ln U = \sum_{i=1}^{n} \beta_i \ln q_i$$

$$m = \sum_{i=1}^{n} p_i q_i$$

- Maximise *u*
- Subject to *m*

Lagrangian

$$\mathfrak{L}(\mathrm{q};\lambda) = \sum_i \beta_i \ln q_i + \lambda \left(m - \sum_i p_i q_i \right)$$

- Differentiate w.r.t q_i and λ

First-Order Conditions

$$\frac{\partial \mathfrak{L}}{\partial q_i} = \frac{\beta_i}{q_i} - \lambda p_i = 0$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda} = m - \sum_{i=1}^{n} p_i q_i = 0$$

Demand function

- Sum FOC 1 and sub into FOC 2

$$q_i = \frac{\beta_i}{\sum_j \beta_j} \frac{m}{p_i}$$

- Demand is homogeneous of degree 0 in prices and income. Consumers aren't fooled by increases in nominal income if prices also rise.

1. The Cobb-Douglas Utility Function:
c. Elasticities and expenditure share

Demand function

$$q_i = \frac{\beta_i}{\sum_j \beta_j} \frac{m}{p_i}$$

Price and income elasticities

Own price elasticity

$$\epsilon_{ii} = \frac{\partial q_i}{\partial p_i} \frac{p_i}{q_i} = -\frac{\beta_i}{\sum \beta_j} \frac{m}{p_i q_i} = -1$$

- Negative so obeys law of demand

Cross-price elasticity

$$\epsilon_{ij} = \frac{\partial q_i}{\partial p_j} \frac{p_j}{q_i} = 0 \qquad i \neq j$$

- Zero so *i, j* are grossly independent

Income elasticity

$$\epsilon_{im} = \frac{\partial q_i}{\partial m} \frac{m}{q_i} = \frac{\beta_i}{\sum \beta_j} \frac{m}{p_i q_i} = 1$$

- Ratio of average to marginal demand function
- Positive so all goods are normal goods

Expenditure Share

$$\frac{p_i q_i}{m} = \frac{\beta_i}{\sum_j \beta_j}$$

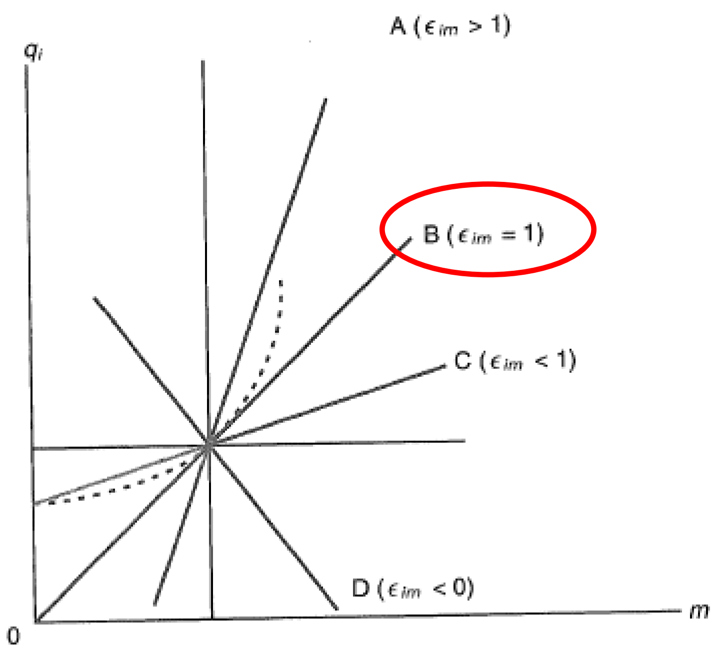
- Constant expenditure share
- Independent of total expenditure – as utility is homothetic

1. The Cobb-Douglas Utility Function:
d. Engel Curve

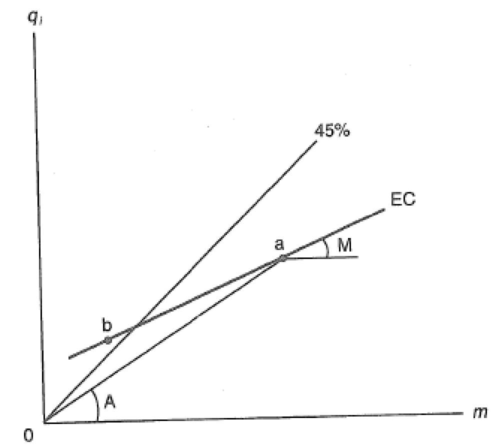
Demand function

$$q_i = \frac{\beta_i}{\sum_j \beta_j} \frac{m}{p_i}$$

Engel Curve



- Straight line
- Passes through the origin so average (A) and marginal (M) function always equal, and income elasticity =1.



Source: Chung, Ch 1.

2. The Stone-Geary Utility Function:
a. The function and its basic properties

	Formula	Intuition
Utility Function	$U = \prod_{i=1}^{n} (q_i - \gamma_i)^{\beta_i}$ $u = \ln U = \sum_{i=1}^{n} \beta_i \ln(q_i - \gamma_i)$ $0 < \beta_i < 1$	• Introduces subsistence level of consumption to Cobb-Douglas (*γ*) • If *γ=0* then this is equivalent to Cobb Douglas
Properties	Monotonic: $\frac{\partial u}{\partial q_i} = \frac{\beta_i}{q_i - \gamma_i} > 0$	• Marginal utility of each good is positive
	Concave utility function $\frac{\partial^2 u}{\partial q_i^2} = -\frac{\beta_i}{(q_i - \gamma_i)^2} < 0$	• Marginal utility of each good is decreasing
	Strongly additive $\frac{\partial^2 u}{\partial q_i \partial q_j} = 0$	• Marginal utility of good *i* is independent of good *j*.
	Not homothetic $\theta^r u \neq \sum_i \beta_i \ln(\theta q_i - \gamma_i)$	• Utility doesn't rise by a scalar if each commodity is multiplied by a scalar, as γ introduces a fixed component of consumption

2. The Stone-Geary Utility Function:
b. Deriving consumer demand

Utility Function and Budget Constraint

$$u = \ln U = \sum_{i=1}^{n} \beta_i \ln(q_i - \gamma_i)$$ • Max *u*

$$m = \sum_{i=1}^{n} p_i q_i \qquad \sum_i \beta_i = 1$$ •S.t. *m*

Lagrangian

$$\mathfrak{L}(\mathrm{q}; \lambda) = \sum_i \beta_i \ln(q_i - \gamma_i) + \lambda \left(m - \sum_i p_i q_i \right)$$

•Differentiate w.r.t q_i and λ

First-Order Conditions

$$\frac{\partial \mathfrak{L}}{\partial q_i} = \frac{\beta_i}{q_i - \gamma_i} - \lambda p_i = 0$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda} = m - \sum_{i=1}^{n} p_i q_i = 0$$

Demand function

•Sum FOC 1 and sub into FOC 2

$$q_i = \gamma_i + \beta_i \frac{m - \sum_i p_i \gamma_i}{p_i}$$

- Consumers first set aside subsistence levels of goods:

$$\sum p_i \gamma_i$$

- Consumers then allocate remaining budget:

$$m - \sum_i p_i \gamma_i$$

in proportion to preferences β (the marginal budget share).

- Food is an example: poor houses spend a greater proportion of income on it (**Engel's Law**)

2. The Stone-Geary Utility Function:
c. Elasticities and expenditure share

Demand function

$$q_i = \gamma_i + \beta_i \frac{m - \sum_i p_i \gamma_i}{p_i}$$

Price and income elasticities

Own price elasticity

$$\epsilon_{ii} = \frac{\partial q_i}{\partial p_i} \frac{p_i}{q_i} = -\frac{\beta_i [1 - (\sum_j p_j \gamma_j - p_i \gamma_i)/m]}{S_i} < 0$$

$$S_i = \frac{p_i q_i}{m}$$

•Obeys law of demand but inelastic ($|\epsilon_{ii}| < 1$)

Cross-price elasticity

$$\epsilon_{ij} = \frac{\partial q_i}{\partial p_j} \frac{p_j}{q_i} = -\frac{\beta_i (p_j \gamma_j / m)}{S_i} < 0 \qquad i \neq j$$

•Less than zero so *i, j* are grossly complements

Income elasticity

$$\epsilon_{im} = \frac{\partial q_i}{\partial m} \frac{m}{q_i} = \frac{\beta_i}{S_i} > 0$$

• Ratio of average to marginal demand function

•Positive so all goods are normal goods

Expenditure

Engel expenditure function

$$p_i q_i = p_i \gamma_i + \beta_i \left(m - \sum_j p_j \gamma_j \right) \qquad i \neq j$$

• This is the amount spent on each good.

• This is a linear function of income and prices.

• Hence, the Stone-Geary function is often called the "linear expenditure system".

•*βi* is the proportion of supernumerary income spent on good *i* $\beta_i = \frac{\partial (p_i q_i)}{\partial N} = \frac{\partial (p_i q_i)}{m}$

Expenditure Share`

$$\frac{p_i q_i}{m} = \frac{p_i \gamma_i}{m} + \frac{\beta_i \left(m - \sum_j p_j \gamma_j \right)}{m} \qquad i \in j$$

• Function of *m* as non-homothetic

3. The Constant Elasticity of Substitution Utility Function:
a. The function and its basic properties

	Formula	Intuition
Utility Function	$$u = \left(\sum_{i=1}^{n} \beta_i q_i^{-\rho}\right)^{-1/\rho}$$ $$\rho = \frac{1-\sigma}{\sigma} < 1$$	• More general form of utility function • Limits elasticity of substitution to a constant σ.
Properties	Monotonic: $$\frac{\partial u}{\partial q_i} = \beta_i u^{1+\rho} q_i^{-(1+\rho)} > 0$$	• Marginal utility of each good is positive
	Concave utility function $$\frac{\partial^2 u}{\partial q_i^2} = \left(\frac{1+\rho}{u}\right) \mathrm{MU}_i \frac{\mathrm{MU}_i q_i - u}{q_i} < 0$$	• Marginal utility of each good is decreasing
	Not additive $$\frac{\partial^2 u}{\partial q_i \partial q_j} \neq 0$$	• Marginal utility of good *i* is not independent of good *j*. However, it is separable (ratio of marginal utilities doesn't depend on third good).
	Homogeneous $$\left[\sum_{i=1}^{n} \beta_i (\theta q_i)^{-\rho}\right]^{-1/\rho} = \theta u$$	• Utility rises by a scalar if each commodity is multiplied by a scalar

3. The Constant Elasticity of Substitution Utility Function: b. Deriving consumer demand

Utility Function and Budget Constraint

$$u = \left(\sum_{i=1}^{n} \beta_i q_i^{-\rho} \right)^{-1/\rho} \qquad \rho = \frac{1-\sigma}{\sigma} < 1$$

$$m = \sum_{i=1}^{n} p_i q_i$$

- Max u s.t. m

Lagrangian

$$\mathfrak{L}(\mathbf{q};\lambda) = \left(\sum_{i=1}^{n} \beta_i q_i^{-\rho} \right)^{-1/\rho} + \lambda \left(m - \sum_{i} p_i q_i \right)$$

- Differentiate w.r.t q_i and λ

First-Order Conditions

$$\frac{\partial \mathfrak{L}}{\partial q_i} = \left(\sum_{j} \beta_j q_j^{-\rho} \right)^{-1/\rho - 1} \beta_i q_i^{-\rho-1} - \lambda p_i = 0$$

$$\frac{\partial \mathfrak{L}}{\partial \lambda} = m - \sum_{i=1}^{n} p_i q_i = 0$$

Demand function

- Take ratio of two FOC 1s, sum over j and rearrange with FOC 2

$$q_i = \frac{\beta_i^{\sigma} p_i^{1-\sigma}}{\sum_{j=1}^{n} \beta_j^{\sigma} p_j^{1-\sigma}} \frac{m}{p_i}$$

- Demand is a linear function of income, as the utility function is homothetic

3. The Constant Elasticity of Substitution Utility Function:
c. Elasticities and expenditure share

Demand function

$$q_i = \frac{\beta_i^\sigma p_i^{1-\sigma}}{\sum_{j=1}^n \beta_j^\sigma p_j^{1-\sigma}} \frac{m}{p_i}$$

Price and income elasticities

Own price elasticity

$$\epsilon_{ii} = -1 + (1-\sigma)S_j \qquad S_i = \frac{p_i q_i}{m}$$

•Obeys law of demand and can choose elasticity

Cross-price elasticity

$$\epsilon_{ij} = -(1-\sigma)S_j$$

- Negative when σ<1: gross complements
- Positive when σ>1: gross substitutes

Income elasticity

$$\epsilon_{im} = 1$$

- Only allows normal goods and Engel curve is linear through the origin.

Expenditure

Engel expenditure function

$$p_i q_i = \frac{\beta_i^\sigma p_i^{1-\sigma}}{\sum_{j=1}^n \beta_j^\sigma p_j^{1-\sigma}} m$$

- Linear in income

Expenditure Share

$$S_i = \frac{\beta_i^\sigma p_i^{1-\sigma}}{\sum_{j=1}^n \beta_j^\sigma p_j^{1-\sigma}} m$$

•Independent of income as utility is homothetic

Week 3 Overview: Utility functions

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3. We use utility functions to derive demand curve, by choosing the mix of goods that maximises utility subject to a budget constraint
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