Leave the Volatility Fund Alone: Principles for Managing Oil Wealth

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Abstract

How should capital-scarce countries manage their volatile oil revenues? Existing literature is conflicted: recommending both to invest them at home, and save them in sovereign wealth funds abroad. I reconcile these views by combining a stochastic model of precautionary savings with a deterministic model of a capital-scarce resource exporter. I show that both developed and developing countries should build an offshore Volatility Fund, but refrain from depleting it when oil prices fall because it cannot be known when, or if, they will rise again. Instead, consumption should adjust and only the interest on the fund should be consumed. To do this I develop a parsimonious framework that nests a variety of existing results as special cases, which I present in four principles: for capital-abundant countries, i) smooth consumption using a Future Generations Fund, and ii) build a Volatility Fund quickly, then leave it alone; and for capital-scarce countries, iii) consume, invest and deleverage, and iv) invest part of the Volatility Fund domestically, then leave it alone.

Keywords: Natural resources, oil, volatility, sovereign wealth fund, precautionary saving, capital scarcity, anticipation.

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1 Introduction

How should capital-scarce, developing countries manage their oil revenues? Current policy advice for developed[1] countries is straightforward: save the revenues offshore in Future Generations and Volatility Funds to smooth expenditure (Davis et al., 2001; Barnett and Ossowski, 2003; Baunsgaard et al., 2012; van der Ploeg and Venables, 2012). However, advice for developing countries is less clear. Some work argues that developing countries should invest their oil revenues domestically (Ramsey, 1928; van der Ploeg and Venables, 2011; IMF, 2012), using a temporary offshore Parking Fund to alleviate absorption constraints (van der Ploeg, 2012; van der Ploeg and Venables, 2013; Venables and Wills, 2016; Araujo et al., 2016). Others recommend saving abroad in a Volatility Fund when oil prices are high, and consuming the principal in this fund to smooth government spending when oil prices fall (van der Ploeg, 2010; van den Bremer and van der Ploeg, 2013; Bems and de Carvalho Filho, 2011; Cherif and Hasanov, 2011 and 2013; Berg et al., 2012; Agenor, 2016). How can we reconcile these seemingly contradictory views? The challenge is to develop a framework that parsimoniously accommodates capital scarcity and volatility. This paper fills that gap.

How natural resources are managed is important. Non-renewable resources dominate nearly a quarter of the world's economies. Of the approximately 200 sovereign states in the world, 130 are endowed with natural resources and 47 are resource-dependent (by the IMF's definition, Baunsgaard et al., 2012). Sixty per cent of resource-dependent economies have a sovereign wealth fund, though they are more common in wealthier nations (SWF Institute, 2016). Many countries manage this wealth poorly: Collier and Goderis (2009) estimate that if commodity exports account for 35% of GDP, a 10% increase in commodity price leads to a 4 to 5% lower long run level of GDP per capita.[2] This "resource curse" has been attributed to de-industrialization (or "Dutch disease", see Corden and Neary, 1982),

[1] Following van der Ploeg and Venables (2011), I draw the distinction between developed and developing countries as a function of their access to capital. Developed countries are capital abundant as they can readily borrow at the world interest rate. Developing countries are capital scarce as they face a premium on their borrowing rate that increases with debt, and can be interpreted as a risk premium (see footnote 13).

[2] There is a large body of literature on the resource curse starting with Gylfason et al. (1999) and Sachs and Warner (2001), and reviewed by van der Ploeg (2011) and van der Ploeg and Poelhekke (2016).

oil price volatility (Ramey and Ramey, 1995; Blattman et al., 2007; Poelhekke and van der Ploeg, 2009; van der Ploeg, 2010), political instability and corruption (Sala-i-Martin and Subramanian, 2003; Acemoglu et al., 2004), and environmental degradation (such as the case of Nauru in Hughes, 2004).

To reconcile the different views on managing oil revenues in developing countries I develop the first model to combine the stochastic literature on precautionary saving with the deterministic literature on capital-scarcity. The model has an optimizing social planner that buys and sells a consumption good at the world price, and sells oil at the world price. I treat oil revenues as a windfall of foreign exchange, because in developing countries oil is extracted using foreign capital and labour, and sold for US dollars, and oil rents dwarf oil's role in domestic production.[3] The revenues are anticipated, as it can take years for to build the rigs, pipelines and processing facilities necessary for production (Arezki et al., 2016; Wills, 2016). Oil revenues are also volatile relative to the consumption good[4], and the exposure to volatility is higher in countries with extensive remaining reserves (e.g. Venezuela, Canada and Iraq) than those nearing depletion (e.g. Brazil, Oman and Brunei; BP, 2016). To do this oil prices are modeled using a random walk without drift, following empirical evidence.[5] Using stochastic optimal control the model can be summarized in two equations, which describe the expected paths of consumption (the Euler equation) and total wealth (the budget constraint), which can be solved analytically.

I find that countries – developed and developing – should manage volatile oil revenues

[3]Resource wealth is often modeled as an exogenous windfall in this way, see van der Ploeg and Venables (2011), (2012) and (2013) and van den Bremer et al. (2016).

[4]Regnier (2007) finds that the prices of crude oil, refined petroleum, and natural gas are more volatile than 95% of products sold by US producers. Van der Ploeg (2011) states that resource revenues are much more volatile than GDP because their supply exhibits a low price elasticity.

[5]There remains considerable debate on whether oil prices are stationary (see, for example Pieschacon, 2012). Results depend on the definition of oil prices, the frequency of the data and the length of the sample studied. In particular, it is important whether the sample includes structural changes in the oil market, like those seen in 1974, 1986 and 2014. This paper is focused on the long horizons over which decisions about sovereign wealth funds are made, which may include many unforeseen changes to oil market structure. I therefore adopt the conservative assumption that oil prices follow a random walk. This follows Engel and Valdes (2000), Bems and de Carvalho Filho (2011) and van den Bremer and van der Ploeg (2013) among others. Alquist et al. (2011) find that real oil prices are best approximated by a random walk beyond six months, and call the random walk without drift a "natural benchmark" for oil prices. Hamilton (2009) also supports approximating oil prices using a random walk without drift, stating "To predict the price of oil one quarter, one year, or one decade ahead, it is not at all naive to offer as a forecast whatever the price currently happens to be."

by precautionarily saving in a Volatility Fund, but the fund's principal should not be depleted when oil prices fall. This is because, when oil prices fall, policymakers cannot know when, or if, they will rise again (formally, oil prices follow a random walk without drift, see footnote 5). When oil prices fall, oil producers are permanently poorer in expectation. Depleting a Volatility Fund's principal is therefore unsustainable, and the government should instead tighten public consumption.

Why, then, should countries build a Volatility Fund at all? The reason, as has previously been identified, is precautionary savings (Leland, 1968; Sandmo, 1970; Zeldes, 1989 and reviews by Carroll and Kimball, 2008, and van den Bremer and van der Ploeg, 2013). Precautionary savings compensates the social planner in the future for bearing the risk of income volatility, by forgoing consumption today to accumulate assets that will generate interest tomorrow. The Volatility Fund is therefore an additional permanent source of income for the social planner, much like the Future Generations Fund, rather than a "buffer" where the principal can be consumed when oil prices are low.[6] Consuming the principal in this way would reduce the income available for future consumption, as oil prices cannot be expected to rise again, and thus be unsustainable. This differs importantly from the advice currently given to resource-rich countries, which is based on building a buffer to "cushion the impact of volatility" (Collier et al., 2010; see also IMF, 2012; van der Ploeg and Venables, 2013).

To build up to this finding I derive four principles for managing resource wealth, which nest a collection of new and existing results for both developed and developing countries.

As already noted, developed countries are currently advised to set up an offshore Future Generations Fund, which replaces the temporary revenue from below-ground oil with permanent revenue from above-ground financial assets.[7] I replicate this baseline advice

[6]The desire for precautionary savings is driven by the social planner's "prudence", which is a characteristics of the third derivative of the utility function. In contrast, the desire for involatile consumption is driven by "risk aversion", a characteristic of the utility function's second derivative (see Carroll and Kimball, 2008). For more details see footnote 11.

[7]This shares similarities with Friedman's (1957) permanent income hypothesis, in the sense that temporary income is saved to smooth consumption over time. However, we make the non-trivial distinction of focusing on an infinitely-lived social planner rather than an individual agent with a finite life. Rather than using the permanent income hypothesis to justify a sovereign wealth fund, Hsieh (2003) uses payments from Alaska's Permanent Fund to test the hypothesis, and confirms it by showing that households

in Principle 1: *“Smooth consumption using a Future Generations Fund”*, by abstracting from both capital scarcity and volatility. I show that the social planner should borrow before the windfall, save abroad in the Future Generations Fund during it, and consume only the permanent income afterwards. The size of the windfall matters: if it is large or long, then more should be borrowed beforehand and less saved during.

As I also note above, developed countries are advised to save precautionarily in an offshore Volatility Fund, and draw down that fund when oil prices fall. Principle 2: *“Build a Volatility Fund quickly, then leave it alone”* presents the first key result of the paper, which argues that only the interest on the Volatility Fund should be consumed, rather than the principal. In addition, I find that building the Volatility Fund should be prioritized in the early years, when the exposure to oil price volatility is greatest. As Volatility Funds should be treated as long term income, they can be invested in a long-term diversified portfolio like a Future Generations Fund, rather than liquid, short-term assets. Spending only the interest on the fund is the approach of Norway’s Government Pension Fund Global (GPFG), which incorporates precautionary savings because spending rises with the size of the fund. These findings are also consistent with Engel and Valdes (2000), who study precautionary savings and find that countries should adjust spending to oil shocks if the costs of doing so are not too large; Landon and Smith (2015), who advocate fixed deposit and withdrawal rates into SWFs; and Pieschacon (2012), who empirically finds that Norway’s approach has shielded the economy from oil price fluctuations, relative to Mexico’s approach of spending oil revenue as it is earned. They differ from Wagner and Elder (2005), who advocate drawing down fiscal stabilization funds, though they focus on business cycles which are more predictable than oil prices.

While developed countries are encouraged to save their resource revenues abroad, developing countries should consume them, invest them in domestic capital, and use them to repay foreign debt (Collier et al., 2010; van der Ploeg and Venables, 2011 and 2013). I replicate this by introducing capital scarcity while abstracting from volatility in Principle 3: *“Consume, invest and deleverage if capital is scarce”*. When oil is discovered,

do smooth anticipated income.

consumption should rise relatively more in capital-scarce than capital abundant countries, because the social planner in the former will be richer in the future. Investment should also be higher, as oil revenues help overcome financing constraints and domestic capital will have a higher realized (and social) rate of return than foreign assets. Finally, foreign debt should be repaid, which helps reduce a debt-elastic cost of borrowing.

In addition, developing countries have been advised to build Volatility (or "Stabilization") Funds to smooth short-term fluctuations in oil prices. The existing view is that the principal in these Funds should be accumulated when oil prices are high, and depleted when they are low (Collier et al., 2010; van der Ploeg and Venables, 2013). In contrast, I show that these Funds should be treated as a source of permanent income rather than a temporary fiscal buffer. I do this by modeling both capital scarcity and volatility for the first time in Principle 4: *"Invest part of the Volatility Fund domestically, then leave it alone"*.[8] Capital-scarce countries should build a smaller Volatility Fund than their capital-abundant neighbours, because they have a lower level of – and thus a higher marginal utility from - consumption.. The savings that are directed to the Volatility Fund should be used to generate permanently higher income, rather than smooth fluctuations. The best way to do this is to invest in high-yielding domestic capital and repay foreign debt – keeping both rates of return in balance. There is no need to focus on liquid short-term assets, because the Fund's principal should not be consumed.

Taken together, these results argue that oil-rich government must show restraint when commodity prices fall. Stabilizing government expenditure is, in itself, insufficient justification for depleting a sovereign wealth fund, because there is no way to know when, or if, oil prices will rise again. Such stabilization would therefore reduce consumption excessively in the future. The better response is to tighten government spending through the annual budget process. Income from Volatility and Future Generations funds will partially insulate the budget from oil price shocks, and the associated reduction in de-

[8]van den Bremer and van der Ploeg (2013) illustrate precautionary savings in a two-period model with capital scarcity. I extend this to many periods, and include investment. Agenor (2016) provides a detailed numerical study of oil price shocks in a medium-sized DSGE model with capital-scarcity, and assumes that the social loss function depends on consumption volatility. In contrast I aim to derive analytical results using a simpler model, based on a welfare-maximising social planner.

mand can be accommodated by an inflation targeting central bank (Gali, 2008). Drawing down a volatility fund to smooth government expenditure might be justified if there are considerable adjustment costs or if monetary policy is constrained, such as by an exchange rate peg or currency union, though this is a topic for future work and is not explicitly studied here.

The rest of this paper proceeds as follows. Section 2 introduces the small, partial equilibrium model at the heart of our analysis, and reduces it to two core equations: the Euler equation and the budget constraint. Section 3 then uses the model to investigate policy in a capital-abundant developed economy, yielding Principles 1 and 2. Section 4 extends this to the case of a capital-scarce developing economy, adding Principles 3 and 4. Section 5 concludes with suggested extensions.

2 Model

I use a stochastic, continuous time, partial equilibrium model where all consumption and output is bought and sold at the world price.[9] The social planner receives exogenous income from oil,[10] with quantity O^i and price $P(t)$, and non-oil production, $Y(K^i)$. The planner chooses how much to consume, $C^i(t)$, how much to invest, $I^i(t)$, in domestic capital, $K^i(t)$, and how much to save in foreign assets, $F^i(t)$. The superscript $i = [A, B, C]$ denotes the three phases following an oil discovery at time $t = 0$: Anticipation, where $O^A = 0$ for $0 \leq t < T_1$; Boom, where $O^B = O$ for $T_1 \leq t < T_2$; and Constant income,

[9] This builds on previous partial equilibrium work, see van der Ploeg 2011, van der Ploeg and Venables, 2012 and 2013, van den Bremer and van der Ploeg, 2013.

[10] In practice recoverable oil reserves are continuously changing as new reserves are discovered, and new technologies invented (eg fracking). These discoveries and inventions are often pro-cyclical, as returns to exploration and R&D increase with the oil price. In this model we focus on the discovery of a fixed quantity of oil in the interests of tractability and clarity, though pro-cyclical exploration and R&D is an important topic for future work.

where $O^C = 0$ for $T_2 \leq t^C$. The planner's problem can be summarised as,

$$
\begin{aligned}
J(F^i, K^i, P, t) &= \max_{C(t)} \left[\int_t^\infty U(C^i(\tau)) e^{-\rho(\tau - t)} d\tau \right] & (2.1) \\
& s.t. \\
dF^i(t) &= (r(F^i(t))F^i(t) + P(t)O^i + Y(K^i(t)) - C^i(t) - I^i(t))dt & (2.2) \\
dK^i(t) &= (I^i(t) - \delta K^i(t))dt & (2.3) \\
dP(t) &= \alpha P(t)dt + \sigma P(t) dZ(t) & (2.4)
\end{aligned}
$$

where $J(\cdot)$ is the value function, $U(\cdot)$ is the utility function, ρ is the rate of time preference, $r(F(t))$ is the interest rate faced by the planner which can depend on the level of assets/debt, δ is the depreciation rate on domestic capital and Z is a Wiener process. I also make four further assumptions about utility, interest rates, output and oil prices.

I assume that utility exhibits constant absolute risk aversion (CARA), $U(C) = \frac{1}{-a} e^{-aC}$.[11] This is necessary for two reasons. First, it makes the effect of volatility on consumption very clear. As the absolute degree of risk aversion - and by extension prudence - is constant, the effect of a given level of volatility on consumption will also be constant. It will not depend on the level of consumption itself. This is useful because the level of volatility faced by the planner will change over time, as oil is extracted and less remains exposed to volatile oil prices. While CARA preferences are a stylized representation of reality, they let me isolate the effects of these changes in volatility. Second, they makes explicit solutions possible, which would not be possible with the popular CRRA preferences (as noted by Kimball and Mankiw, 1989, Merton, 1990, and Kimball, 1990). [12]

[11]Precautionary savings plays an important role in the results that follow. This requires a per period utility function that is increasing and concave in consumption and has a positive third derivative, and which yields a savings function that is increasing and convex in available savings plus income. Both CRRA and CARA preferences exhibit these characteristics (Huggett, 2004; Huggett and Vidon, 2002).

[12]There is a trade off between using CRRA and CARA preferences. Consumption is affected by the size of a volatile income stream under CARA, but by the share in total wealth of a volatile income stream under CRRA. While CRRA preferences might match individual behavioral data more closely (Merton, 1969), they exhibit diminishing absolute risk aversion which makes the effects of volatility on consumption less clear. This is because the effect of oil price volatility could fall for two reasons: because oil is being extracted, or because consumption is rising. CRRA would also only allow numerical solutions to our problem, because explicit solutions are only possible if oil volatility is constant as a share of wealth, as discussed in Merton (1990), Chang (2004) and van den Bremer et al. (2016). In contrast, CARA preferences may be more stylized, but they give a much clearer role for volatility and allow for analytical

Interest rates can have a linear premium on debt to capture capital scarcity in developing countries, see equation 2.5.[13] If the country is capital abundant or a net lender, $F(t) \geq 0$, then it will borrow and lend freely at the constant[14] world interest rate, $\omega = 0$; but if it is capital-scarce and indebted, $F(t) < 0$, then its cost of borrowing will rise according to $\omega > 0$.[15]

$$r(F(t)) = \begin{cases} r - \omega F(t) & \text{for } F(t) < 0 \\ r & \text{otherwise} \end{cases} \tag{2.5}$$

A single internationally traded good is produced using constant technology, capital and a fixed supply of labour, $Y(K(t)) = AK(t)^{\alpha}\bar{L}^{1-\alpha}$ where $\bar{L} = 1$, and sold at a constant world price, $P^* = 1$. This is the same as the good consumed by the social planner, $C^i(t)$.

Oil prices follow a geometric Brownian motion with zero drift, $\alpha = 0$ (see empirical evidence in footnote 5). Setting the drift of oil prices to zero makes the analysis simpler and ensures that the present value of the income stream is finite. As price shocks are persistent, a shock today will affect the price in all future periods. Positive or negative shocks are equally likely at any point in time.[16]

Dropping superscripts for simplicity, the Hamilton-Jacobi-Bellman equation for the

solutions, so are more suitable for this analysis.

[13]While most countries have access to international capital markets, developing countries must pay a considerable risk premium on borrowing. There are a number of factors that influence the risk premium on borrowing including political stability, monetary regime, currency of debt issue, etc. This paper follows van der Ploeg and Venables (2011) who find empirical evidence that the log of annual average bond spreads is increasing in the ratio of public and publicly guaranteed debt to gross national income, PPG/GNI. They also do not find evidence for natural resource discoveries alone reducing the cost of borrowing. A debt premium on interest rates is also commonly used to remove unit-roots from small-open economy models (Schmitt-Grohe and Uribe, 2003).

[14]In practice sovereign wealth fund asset prices are also volatile (see van den Bremer et al., 2016), but are typically less so than oil prices which justifies this simplification.

[15]For example, in 2013 Ghana issued a ten-year, dollar denominated bond yielding 8%, whilst ten-year US treasuries yielded 2.5%.

[16]Allowing for drift in oil prices, so long as $\alpha < r$, will alter the intertemporal path of consumption in expectation, but will not change the key results on consuming only the interest from the Volatility Fund in Sections 3.2 and 4.2. The key results would change if oil prices are assumed to mean revert, as oil price shocks would only be temporary. There remains some debate over whether oil prices are mean reverting in practice (see footnote 5). I adopt the more conservative assumption that they do not, and so when the oil price falls one does not know when, or if, they will rise again. This is consistent with long-horizon forecasts of the oil price, see Alquist et al. (2011).

problem in 2.1 to 2.4 is,

$$\max_{C}\left[U(C(t))e^{-\rho t}+\frac{1}{dt}E_t[dJ(F,K,P,t)]\right] = 0 \tag{2.6}$$

where the second term can be expressed as follows using Ito's lemma,

$$\begin{aligned}\frac{1}{dt}E_t[dJ(F,K,P,t)] &= J_F(r(F)F+PO+Y(K)-C-I)\\ &\quad +J_K(I-\delta K)+J_t+\tfrac{1}{2}J_{PP}\sigma^2P^2\end{aligned} \tag{2.7}$$

The first order conditions with respect to consumption, investment, foreign assets, and domestic capital are summarised in the following two equations, derived in Appendix A:

$$\frac{E_t[dJ_F]/dt}{J_F} = -(r-2\omega F) \tag{2.8}$$

$$r-2\omega F = \alpha AK^{\alpha-1}-\delta \tag{2.9}$$

Equation 2.8 describes the evolution of the marginal utility of foreign assets. Equation 2.9 shows that the social planner optimally allocates total assets to equate the marginal benefit of repaying foreign debt (reducing both the stock of debt and the interest paid on it), with the marginal benefit of investing in domestic capital (boosting output). So, if consumption differs from income at any time, both domestic capital and foreign assets should increase or decrease together. Overall domestic capital increases more rapidly than foreign assets, so that its share in total assets, $\bar{S}=F+K$, will rise.

Finding an explicit solution for the value function is not straight-forward because of the risk premium on debt, the non-linear production function and the varying exposure to oil price volatility as oil is extracted. In the next section I explicitly solve the value function when oil prices are certain, which is only possible with CARA preferences. When oil prices are uncertain I will take a different tack.

Rather than explicitly solve for the value function, a lot of intuition can still be gained by examining the expected dynamics of consumption and total assets, $\bar{S}=F+K$; given in equations 2.10 and 2.11 and derived in Appendix A. They are expressed in terms of linear deviations from a steady state, $S = \bar{S} - S^*$, where capital scarcity is overcome and oil income is exhausted, $F^* = 0$ and $O^* = 0$,

$$\frac{1}{dt}E[dC^i(t)] = (r - \rho - \frac{2\omega}{a} f S^i(t)) + \frac{1}{2} a P(t)^2 C_P^i(t)^2 \sigma^2 \tag{2.10}$$

$$\frac{1}{dt} dS^i(t) = rS^i(t) + P(t)O^i - C^i(t) + C^* \tag{2.11}$$

The Euler equation in 2.10 describes the expected dynamics of consumption. The first term describes the typical trade-off between consuming today, or saving at rate r for future consumption, which is discounted at rate ρ. The desire to save also depends on total assets through the risk premium on debt, $F^i(t) = fS(t)$, based on a linear approximation around the steady state, $f = (1 + \frac{\omega}{\alpha(1-\alpha)}(\frac{\alpha}{r+\delta})^2)^{-1}$. The second term describes how oil price volatility affects the expected change in consumption. Higher volatility, σ, delays consumption from today until tomorrow, in line with precautionary savings. The degree of precautionary savings depends on the marginal propensity to consume from a change in the oil price, $C_P^i \equiv \partial C^i(t)/\partial P(t)$, which in turn depends on the size of the remaining windfall.

The budget constraint in 2.11 describes the expected dynamics of total assets. This uses equation 2.9, and is linearised around the steady state level of assets, S^*.[17] The term $rS^i(t)$ describes the rate of return to total assets near this steady state, which can be split into two components (see Appendix A). The first is the rate of return on foreign assets, rf. The second is the marginal product of capital less depreciation, $(Y'(K^*) - \delta)(1 - f) = r(1 - f)$.[18]

The rest of this paper will involve using these two equations to demonstrate the four principles, both explicitly and with the use of diagrams.

[17] Linearising does not remove the effect of the risk premium, ω. It still appears in the Euler equation, so the incentive to save, $\dot{C}^i(t) > 0$, still increases with the level of debt, $F^i(t) = fS(t)$.

[18] I do not assume that the share of foreign assets and domestic capital is constant, but rather that the share of capital in total assets increases linearly, rather than non-linearly as in equation 2.9

3 Oil discoveries in developed countries

3.1 The Future Generations Fund when capital is abundant

When capital is abundant, $\omega = 0$, and prices are certain, $\sigma = 0$, the dynamic equations 2.10 and 2.11 collapse to equations 3.1 and 3.2. If interest rates are constant and goods can be freely bought and sold at the world price then capital will be constant, $K = K^* = (\frac{r+\delta}{\alpha A})^{1/(\alpha-1)}$ from equation 2.9. Any change in total assets will thus be driven by foreign assets, so $f = 1$ and $S^i(t) = F^i(t)$, which are focused on in equation 3.2.[19]

$$\dot{C}^i(t) = \tfrac{1}{a}(r-\rho) \tag{3.1}$$

$$\dot{F}^i(t) = rF^i(t) + PO^i - C^i(t) + C^* \tag{3.2}$$

Solving this system gives the time path of consumption and assets. Setting $r = \rho$; taking the time derivative of 3.2 and substituting in 3.1 gives the single differential equation, $\ddot{F}^i(t) - r\dot{F}^i(t) = 0$, with the general solution,

$$C^i(t) = ra_2^i + PO^i + C^* \tag{3.3}$$

$$F^i(t) = a_1^i e^{rt^i} + a_2^i \tag{3.4}$$

where the parameters a_1^i and a_2^i depend on the phase of the oil boom: $i = [A, B, C]$. They are found by requiring that i) each phase begins with the assets at the end of the last phase, $F(0) = a_1^A + a_2^A$, $F^A(T_1) = a_1^B + a_2^B$ and $F^B(T_2) = a_1^C + a_2^C$, and ii) consumption moves smoothly between phases. Assets are therefore constant at the end of the Boom to satisfy the transversality condition. Moving recursively through each phase gives,

$$\begin{aligned} a_1^C &= 0 &;\quad a_2^C &= F(T_2) \\ a_1^B &= POe^{-r(T_2-T_1)}/r &;\quad a_2^B &= F(T_1) - a_1^B \\ a_1^A &= -PO(e^{-rT_1} - e^{-rT_2})/r &;\quad a_2^A &= F(0) - a_1^A \end{aligned} \tag{3.5}$$

[19] These equations hold without approximation. Note the change in notation from $\frac{1}{dt}E[dC^i]$ to $\dot{C}^i(t)$, to emphasise that the expected change in each variable is deterministic.

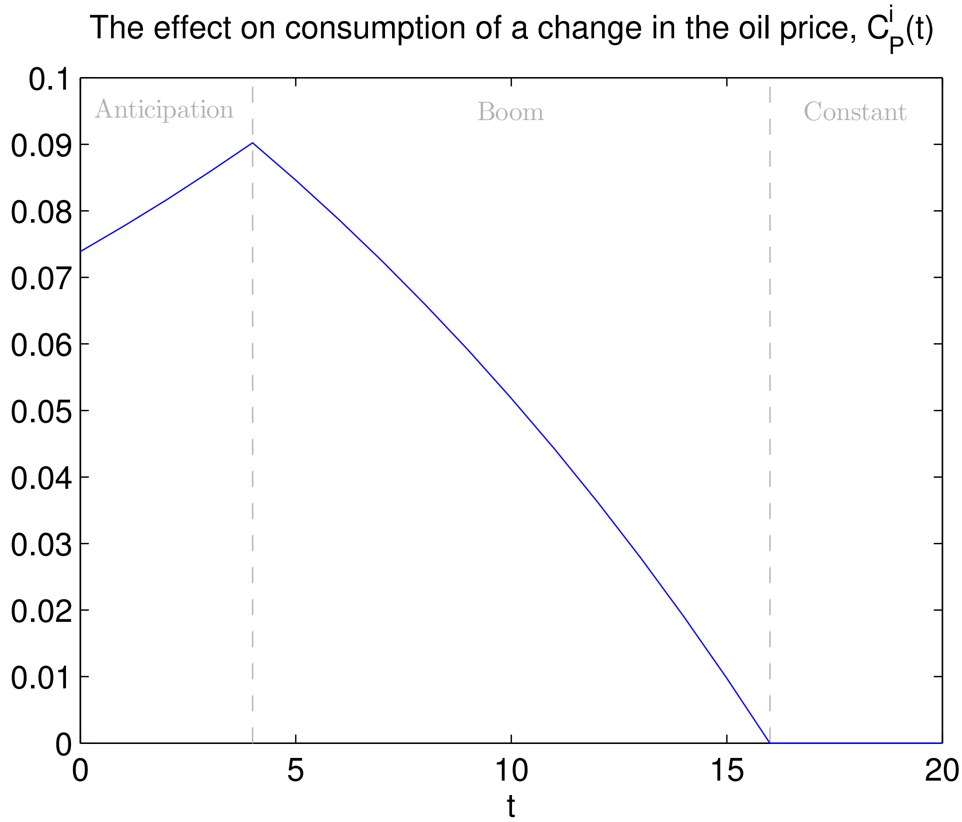

Figure 3.1: Consumption is most affected by oil price changes at the start of the Boom, when the present value of future revenues is highest. Consumption is less affected as oil is extracted and less remains exposed to price changes. Based on calibration in Appendix C.

The marginal propensity to consume from a change in oil prices, $C_p^i(t)$, depends on the remaining value of oil and is found by combining equations 3.3 and 3.5, see equation 3.6.[20] It shows that oil price volatility has an increasing effect on consumption during the Anticipation phase, which falls as oil is depleted during the Boom, see Figure 3.1.

$$\begin{aligned} C_P^C(t) &= 0 && \text{for} \quad T_2 \leq t \\ C_P^B(t) &= O(1 - e^{-r(T_2-t)}) && \text{for} \quad T_1 \leq t < T_2 \\ C_P^A(t) &= O(e^{-r(T_1-t)} - e^{-r(T_2-t)}) && \text{for} \quad 0 \leq t < T_1 \end{aligned} \tag{3.6}$$

This gives the first principle of managing resource windfalls,

Principle 1. ***"Smooth consumption using a Future Generations Fund"***

i) A social planner facing a certain but temporary windfall, which can freely borrow

[20]It is difficult to derive in a stochastic setting, so the following sections use the deterministic case as an approximation (see also van den Bremer and van der Ploeg, 2013).

at the world rate of interest, should smooth consumption based on the rates of return and time preference, $\dot{C}^i(t) = \frac{1}{a}(r - \rho)$ *for all* t*.*

ii) Consumption will be a constant proportion of total wealth: the sum of foreign assets, oil wealth and steady state consumption, $C^i(t) = rW(t)$ *where* $W(t) = F(t) + V(t) + \frac{1}{r}C^*$ *and* $r = \rho$*.*

iii) This will involve borrowing before the windfall, saving during the windfall in a Future Generations Fund, and consuming the interest earned after the windfall, $\dot{F}^A(t) < 0, \dot{F}^B(t) > 0, \dot{F}^C(t) = 0$*.*

iv) Less will be saved from a long windfall, and more borrowed beforehand, $\frac{\partial}{\partial T_2}\dot{F}^i(t) < 0$ *for* $t < T_2$*.*

v) Investment will be independent of any oil discovery if capital is abundant and goods are freely traded, consistent with the Fisher separation theorem, $K = K^*$*.*

Proof. See Appendix 1. □

The first principle makes the case for establishing a sovereign wealth fund, to replace the assets below the ground with assets above it, see the blue lines in Figure 3.2. In capital-abundant countries oil discoveries should not affect domestic investment, because all profitable projects should already be financed by debt (the Fisher, 1930, separation theorem). Therefore, savings should be directed into a Future Generations fund abroad. This is similar to Hartwick (1977), who first argued that revenues from exhaustible resources should be invested in above-ground assets. However, he ignored access to foreign assets so the only asset available to him was domestic capital. It also replicates similar results in van der Ploeg and Venables (2011) and (2012), Venables and Wills (2016) and van den Bremer et al. (2016) among others.

Consumption will be smooth, but may not be constant. Part i) shows that if the rate of interest is higher than the discount rate (e.g. through rapid technological progress that raises the rate of return on saving), $r > \rho$, then consumption will grow. The amount depends on the degree of intertemporal substitution, a, which is also the coefficient of

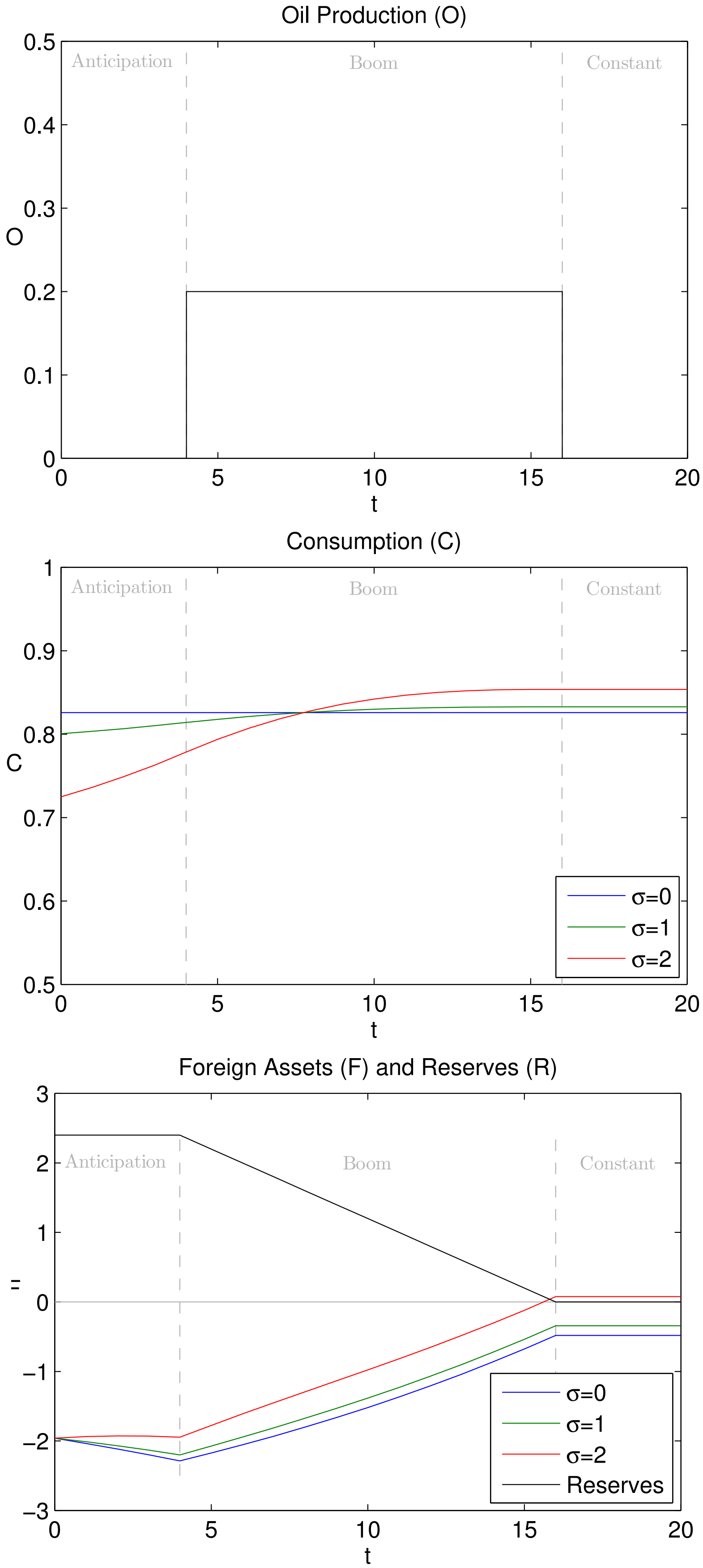

Figure 3.2: An oil boom should lead to borrowing before it begins, saving during, and consuming interest after it ends. Volatility will lead to additional precautionary savings (from equations, 3.3, 3.4, 3.9, and 3.10).

absolute risk aversion.[21] In contrast if the social planner is impatient (the discount rate is high), then more will be consumed in the early years of the windfall. This may happen if they value future consumption less than current consumption.[22]

Longer windfalls will involve less saving. This is because the income produced by a long windfall will be closer to its permanent income. In the limit, if a constant windfall becomes permanent then it makes sense to consume all income as it is received.

The effects of an oil discovery can also be illustrated using the phase diagram in Figure 3.3. The blue lines are the same as those in Figure 3.2, but directly compare the movement of consumption and foreign assets. The steady state line starting in the lower-left corner illustrates where consumption exactly equals non-oil income before and after the windfall, $\dot{F}^{A,C} = 0$ and $O^{A,C} = 0$ in equation 3.2. The economy starts on this line, at X_0. On discovering oil consumption jumps up to X_0^A as households become wealthier. Consumption is higher than non-oil income, so assets are consumed and the economy moves to point X_1at the beginning of the boom, $t = T_1$. During the boom households also receive oil income, so the steady state line moves up to the dotted line, $\dot{F}^B = 0$ and $O^B > 0$ in equation 3.2. Consumption is below this level, so assets are accumulated until the end of the boom, $t = T_2$. Consumption will have been chosen so that the economy lands on the initial steady state line at this time, point X_2.

3.2 The Volatility Fund when capital is abundant

When capital is abundant, $\omega = 0$, and prices are volatile, $\sigma > 0$, all profitable projects will already be financed and domestic capital again remains constant. So, I again focus on foreign assets, $f = 1$, and the dynamic equations 2.10 and 2.11 collapse to equations

[21] This is a result of the constant absolute risk aversion utility function. An alternative is to use Epstein-Zin (1989) preferences, which separate intertemporal substitution from risk aversion.

[22] In practice policymakers' discount rates can be increased by concerns of being removed from office (see also Venables and Wills, 2016).

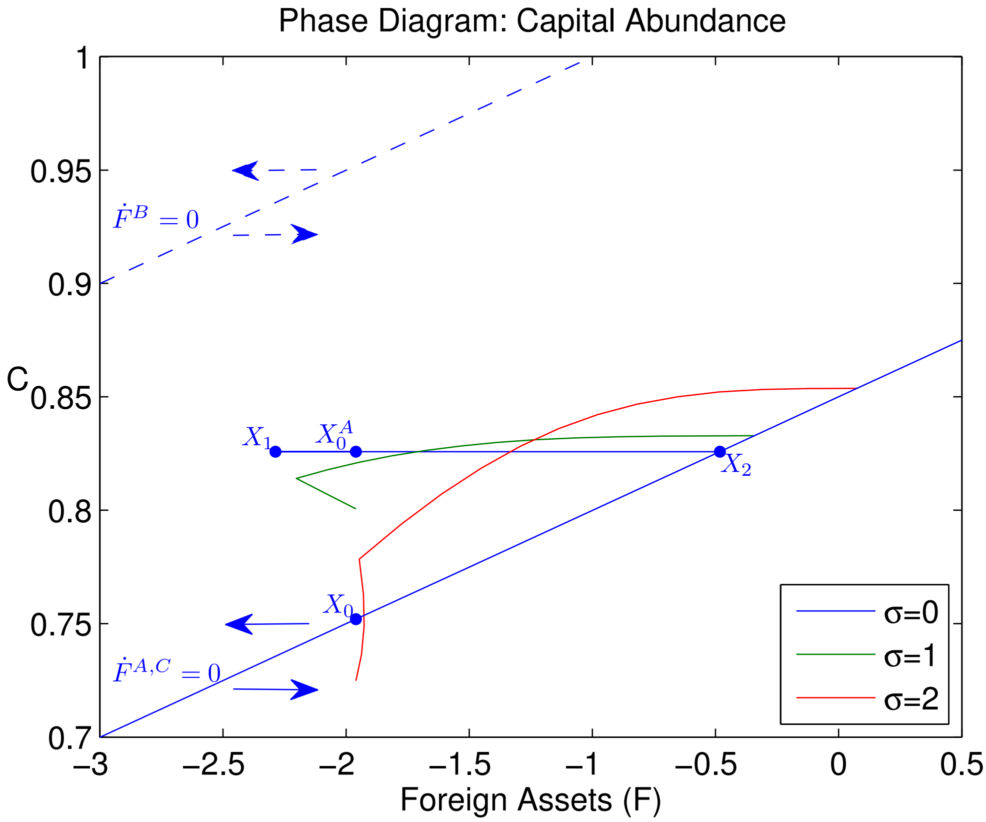

Figure 3.3: An anticipated windfall in a capital-abundant economy. Before and after the windfall the movement of consumption and assets is dictated by the solid blue line and arrows, $\dot{F}^{A,C} = 0$. During the windfall it is dictated by the dashed blue line and arrows, $\dot{F}^B = 0$. If prices are certain the economy will follow the path $[X_0, X_0^A, X_1, X_2]$. If prices are volatile (green and red), then consumption will also rise over time.

3.7 and 3.8,

$$\dot{C}^i(t) = \frac{1}{a}(r - \rho) + \frac{1}{2}aP(t)^2 C_P^i(t)^2 \sigma^2 \quad (3.7)$$

$$\dot{F}^i(t) = rF^i(t) + P(t)O^i - C^i(t) + C^* \quad (3.8)$$

These two equations describe the *expected* dynamics of consumption and assets, and so I treat them as deterministic. The effects of volatility are captured in the second term of the Euler equation, 3.7. Volatility increases the change in consumption, all else being equal, giving rise to precautionary savings in the near term, to fund higher consumption in the future. The volatility adjustment decreases as oil is extracted, $C_P^i(t) \to 0$ in equation 3.6.

The paths of consumption and assets are found by solving this system of equations, which can be summarised by a single differential equation $\ddot{F}^i(t) - r\dot{F}^i(t) = \frac{1}{2}aP^2\sigma^2 C_P^i(t)^2$, as before.[23] The general solution is,

$$F^i(t) = a_1^i e^{rt^i} + a_2^i + F_V^i(t) \quad (3.9)$$

$$C^i(t) = ra_2^i + PO + Y + C_V^i(t) \quad (3.10)$$

where $C_V^i(t) = rF_V^i(t) - \dot{F}_V^i(t)$. To aid interpretation I choose a "particular solution", $F_V^i(t)$, so that a_1^i, a_2^i are the same as before, in equation 3.5. This lets us split the assets accumulated by the social planner into two funds. The Future Generations Fund, $F_{FG}^i(t) = a_1^i e^{rt^i} + a_2^i$, holds the funds that are used to smooth consumption over time, in accordance with principle one. It need not be positive, such as borrowing during the Anticipation phase of a windfall. The Volatility Fund, $F_V^i(t)$, then captures all the funds that are accumulated in response to oil price volatility. The volatility fund will begin at zero before the oil discovery is announced, $F_V^A(0) = 0$, and will always be positive in expectation, $F_V^i(t) \geq 0$ for all t. This brings us to the second principle of managing resource wealth,

Principle 2. ***"Build a Volatility Fund quickly and then leave it alone"***

Build a Volatility Fund quickly:

i) A social planner facing a volatile temporary windfall, who can freely borrow at the world rate of interest, should engage in precautionary savings and build up a "Volatility Fund", in addition to the "Future Generations" fund in principle 1, $F_V^i(t) > 0$ *for all* $t > 0$*.*

ii) The Volatility Fund should receive relatively more savings during the early years of the windfall, including during the Anticipation phase, $\dot{F}_V^A(t), \dot{F}_V^B(t) > 0$ *and* $\ddot{F}_V^A(t), \ddot{F}_V^B(t) < 0$ *for* $t < T_2$*.*

Then leave it alone:

[23] This is a is a non-homogeneous, second-order linear differential equation. If $F^i(t)$ and $F_V^i(t)$ are both solutions, then $\ddot{f}^i(t) - r\dot{f}^i(t) = 0$ for $f^i(t) \equiv F^i(t) - F_V^i(t)$ and the problem collapses to that discussed in section 3.1. The "particular solution" $F_V^i(t)$ can be found by the methods of undetermined coefficients or variation of parameters (see Robinson, 2004).

iii) The assets in the Volatility Fund should not be consumed in the presence of a persistent negative shock to the oil price. Only interest should be consumed, $\frac{\partial}{\partial P_0} F_V^i(0) = 0$.

iv) As oil is extracted, the need for a Volatility Fund will diminish, $\dot{C}^B(t) \to 0$ *as* $t \to T_2$.

v) Any funds remaining in the Volatility Fund should be saved, and only the permanent income consumed, $\dot{F}_V^i(t) \geq 0$ *for all* $t \leq T_2$.

Proof. See Appendix 2. □

Principle two begins by stating that the Volatility Fund should be prioritised at the start of the windfall, when oil price exposure is greatest. Consumption from an oil windfall will depend on two effects: the wealth effect and the precautionary effect. The wealth effect increases consumption, as an oil discovery increases lifetime wealth. The precautionary effect reduces consumption, as oil also makes income more volatile.

The Volatility Fund should be prepared before oil production even begins. During the Anticipation phase the Future Generations Fund will go into debt, borrowing against future income because of the wealth effect. This will be offset by the Volatility Fund, built because of the precautionary effect as illustrated in Figure 3.4. In practice this may involve simply borrowing less in the Future Generations Fund, rather than borrowing in one fund, whilst saving in another. The volatility effect is strongest near the start of the windfall when its present value is highest, as seen in Figure 3.1. If the windfall is particularly long or volatile then CARA preferences permit the counter-intuitive result that consumption can fall below its pre-oil level, as seen in red in Figure 3.3. This is a side-effect of using CARA preferences, and follows from two particular characteristics of them. First, under these preferences the effect of a given level of volatility on consumption is constant, regardless of the level of consumption. Second, CARA preferences also do not exclude the (nonsensical) possibility of consumption falling below zero, $C < 0$.[24] So, if the oil price is very volatile then the volatility effect may outweigh the wealth effect and

[24] This can only be prevented by choosing an appropriate values for the parameters and the starting level of consumption.

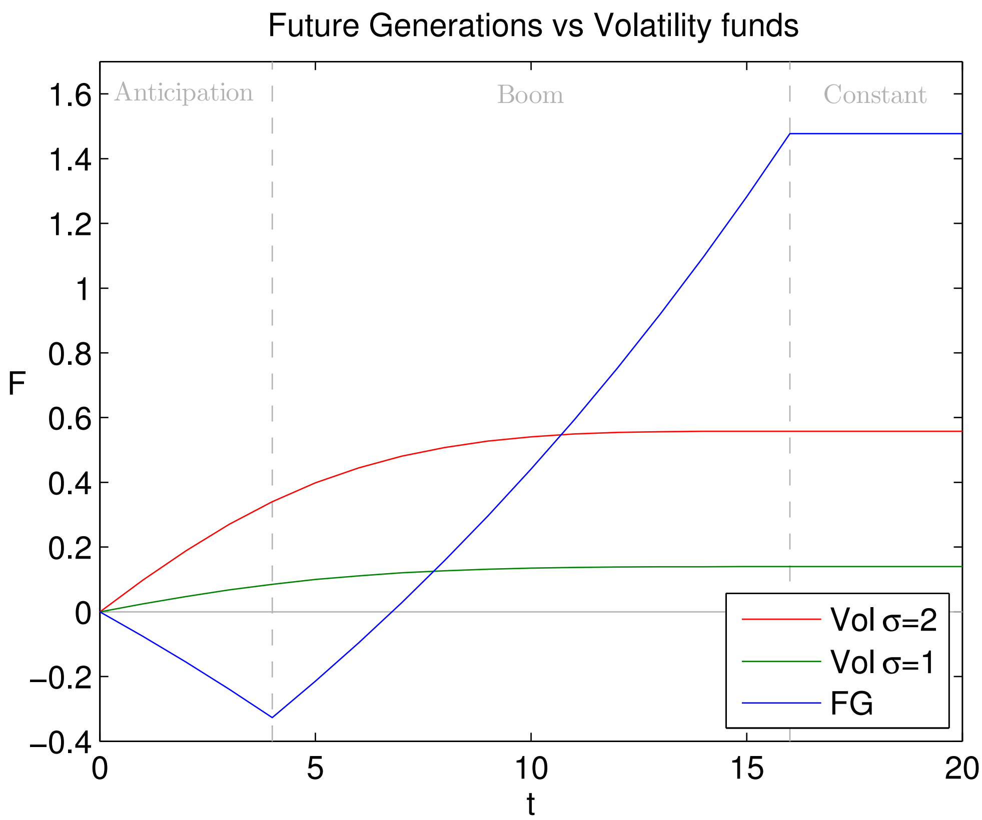

Figure 3.4: The Future Generations and Volatility Funds from the beginning of each stage of the windfall. The Volatility Fund will receive relatively more savings during the early years of the windfall, and may receive more savings in absolute value if the windfall is particularly long or volatile.

marginal consumption from an oil discovery can be negative. This would not occur under other utility functions (such as CRRA). However, the assumption of CARA preferences is crucial in making the model tractable, as discussed in Section 2.

Both the Volatility and the Future Generations Funds should accumulate during oil production. The Volatility Fund will receive relatively more at the start of the windfall, when the exposure to oil price volatility is highest. If the windfall is long or volatile then the Volatility Fund may receive more than the Future Generations Fund in absolute value. The need to save against future shocks will then fall as oil is extracted. After the windfall both the Volatility and Future Generations Funds can be combined, and the interest they earn should finance consumption in perpetuity.

Principle two also emphasises that the Volatility Fund should be treated as a permanent source of income, rather than as a “shock absorber”. In practice it may be tempting for policymakers to consume the Volatility Fund's principal when oil prices are low. However, oil price shocks are very persistent, so if prices are low today then they are also expected to be low in the future (see footnote 5), so the oil exporter is permanently poorer in expectation. Therefore, this principle shows that when the oil price falls the optimizing social planner should reduce its planned path for future consumption. [25] Why have a Volatility Fund at all then? Because the permanent income it generates will allow consumption to be higher than it would otherwise, "compensating" the social planner in the future for bearing income risk. This is because the social planner prefers certain to uncertain income according to Jensen's inequality. These results are illustrated in the example in Figure 3.5, which compares optimal consumption and assets with and without a Volatility Fund, after a negative shock to the oil price.

The need for a Volatility Fund will diminish over time, but its principal should not be consumed. The exposure to oil price volatility falls as oil is extracted. Policymakers in practice may see this as an opportunity to consume the fund once the threat of volatility has passed. However, as the Volatility Fund is designed to generate income, rather than act as a buffer, it can be treated very similarly to the Future Generations Fund and possibly even combined.[26]

The role of the Volatility Fund is also illustrated in the green and red lines of the phase diagram in Figure 3.3, and the time paths in Figure 3.2. Volatility causes consumption to jump less when oil is discovered ($\sigma = 1$) and can even fall ($\sigma = 2$), because of the precautionary effect. Consumption will then rise steadily over time, both before and during the windfall, according to equation 3.7. The Future Generations Fund still goes into debt before the windfall, and accumulates during it. This is offset by the Volatility

[25] This result relies on oil prices following a random walk. If they can be reliably forecast to mean revert then there could be a case for depleting the Volatility Fund's principal while prices are low. In that case the analysis would be more like Wagner and Elder (2005), who use a sovereign wealth fund to smooth business cycles, which are more predictable than oil prices.

[26] This is very similar to the way that Norway's Government Pension Fund Global (GPFG) is managed in practice. Norway's *handlingsregelen* ("budgetary rule") dictates that only a fixed proportion of the Fund's assets can be consumed in any year (previously 4%, tightened to 3% in 2017; see van den Bremer et al., 2016). Consumption is therefore initially low but rises as assets are accumulated, which is consistent with the volatility case in Figure 3.2. The additional savings from consuming less than the steady state level during oil extraction can be thought of as accumulating in a "Volatility Fund" which is incorporated fungibly into the GPFG. When oil is exhausted these additional savings will finance permanently higher consumption in Norway.

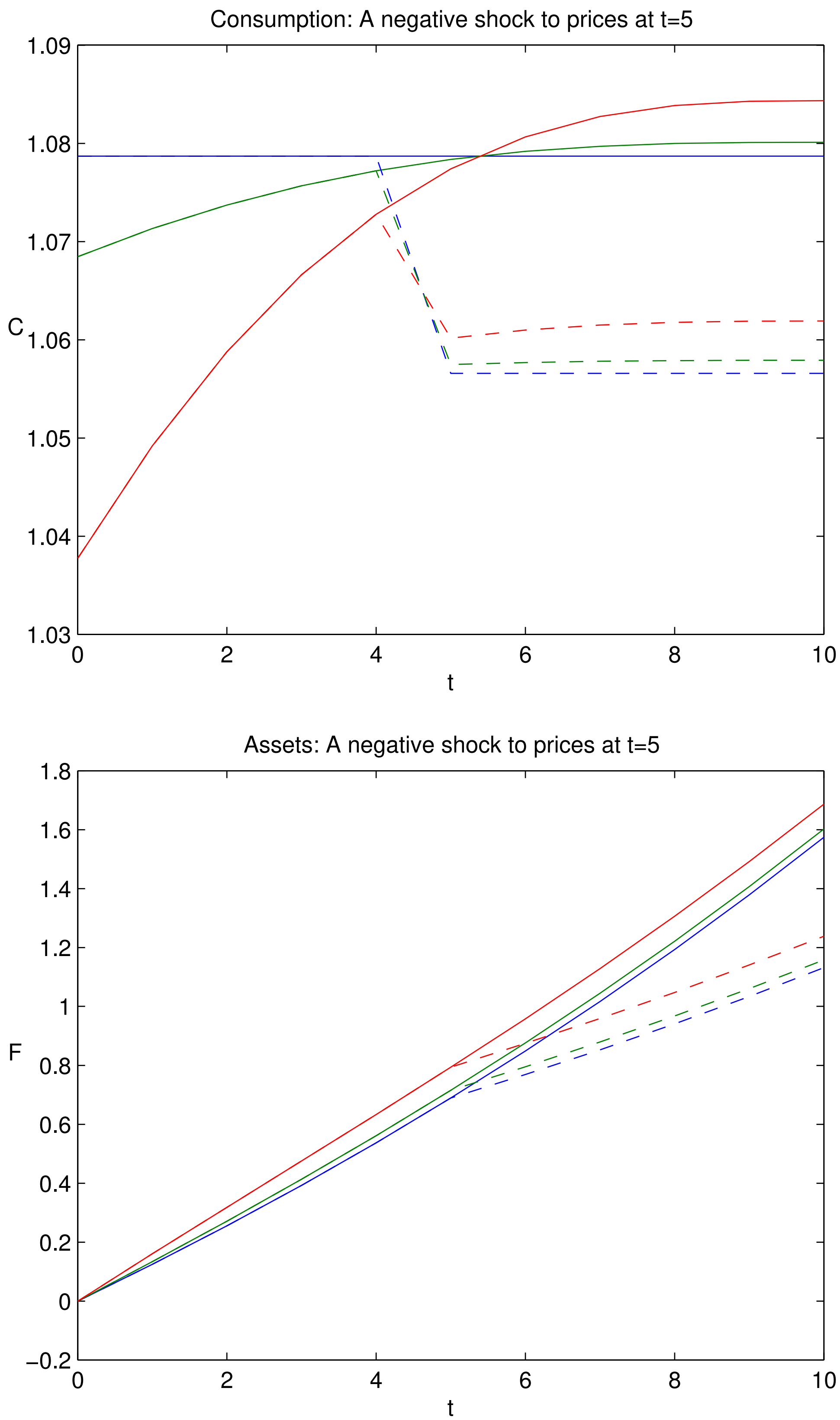

Figure 3.5: Building up a Volatility Fund buffers consumption against negative price shocks. Each line illustrates the expected path of consumption and assets before (solid) and after (dotted) an unanticipated shock to the oil price from $P = 1$ to $P = 0.5$ at $t = 5$, if the planned assumes ex ante there is no price volatility ($\sigma = 1$, blue), low volatility ($\sigma = 1$, green) and high volatility ($\sigma = 2$, red).

Fund, which grows throughout. At the end of the windfall the economy ends on the steady state line, $\dot{F}^{A,C} = 0$, above X_2. The extra assets from the Volatility Fund will then finance higher consumption in perpetuity.

4 Oil discoveries in developing countries

4.1 Consumption, investment and deleveraging when capital is scarce

I consider two types of windfall: small and large. Small windfalls will not overcome capital scarcity before they are exhausted (following the characterisation in van der Ploeg and Venables, 2011). Large booms will, so the economy will begin to behave like those discussed in Section 3 before all oil is depleted.

4.1.1 Small oil discoveries

If the oil discovery is small then the social planner must contend with capital scarcity both during and after the windfall. Consumption and assets will follow equations 4.1 and 4.2, from equations 2.10 and 2.11 with $\sigma = 0$ and $\omega > 0$, for the duration of the windfall. When capital is scarce, repaying debt will reduce the cost of borrowing. Domestic capital will therefore be linked to foreign assets by the asset allocation condition, 2.9. This means that the share of capital in total assets will grow over time as capital is built, so $f \neq 1$ and $F(t) = \frac{f}{1-f}(K(t) - K^*)$ as illustrated in Figure 4.1.

$$\dot{C}^i(t) = r - \rho - \tfrac{2\omega}{a} f S^i(t) \tag{4.1}$$

$$\dot{S}^i(t) = rS^i(t) + PO^i - C^i(t) + C^* \tag{4.2}$$

The paths of consumption and assets are found by solving equations 4.1 and 4.2. Extending the approach in section 3.1, these equations are summarised by a single differential equation, $\ddot{S}^i(t) - r\dot{S}^i(t) - \frac{2\omega}{a} f S^i(t) = 0$. The general solution is as follows, where t^i is

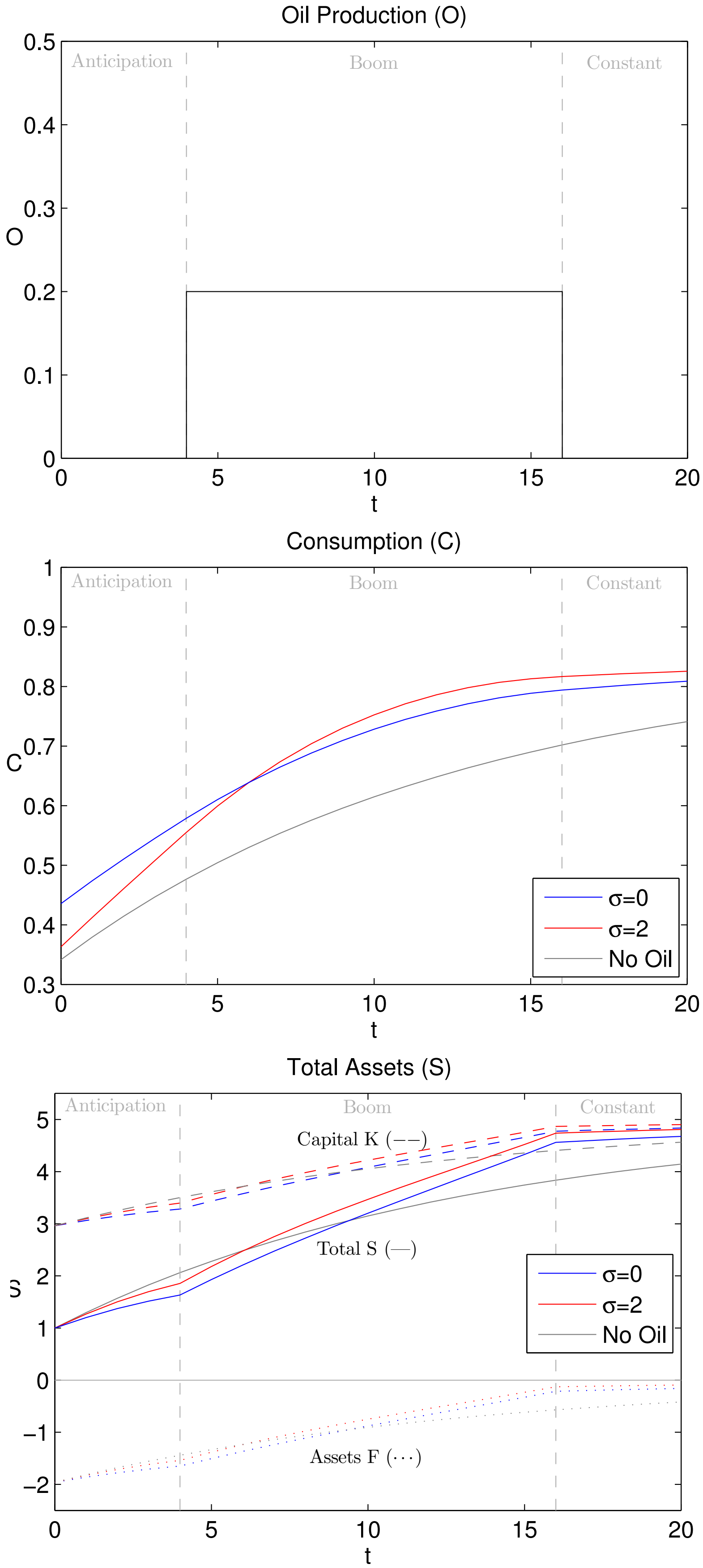

Figure 4.1: An oil boom will accelerate development if capital is scarce. Before the boom both capital and assets will fall, to smooth consumption. During the boom both accumulate quickly, financing an increase in consumption. Volatility leads to more savings (from equations 4.3, 4.4, 4.10, and 4.11).

measured from the beginning of each phase and, $\lambda_j = \frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 + 8\omega f/a}$, are the roots of the characteristic equation with $\lambda_1 < 0$ and $\lambda_2 > 0$,

$$S^i(t^i) = c_1^i e^{\lambda_1 t^i} + c_2^i e^{\lambda_2 t^i} \tag{4.3}$$

$$C^i(t^i) = c_1^i (r - \lambda_1) e^{\lambda_1 t^i} + c_2^i (r - \lambda_2) e_2^{\lambda_2 t^i} + PO^i + C^* \tag{4.4}$$

The coefficients, c_1^i and c_2^i, are found by imposing conditions at the beginning and end of each phase, $i = [A, B, C]$. Each phase begins with the assets left at the end of the last, $S(0) = c_1^A + c_2^A$, $S^A(T_1) = c_1^B + c_2^B$ and $S^B(T_2) = c_1^C + c_2^C$. During the Constant income phase, consumption and assets must stay on the stable saddle path, so $c_2^C = 0$. During the Boom phase consumption must be chosen so that the windfall ends with consumption and assets on the stable saddle path, $C^B(T_2) = (r - \lambda_1)S^B(T_2) + Y$. Recursively, consumption during the Anticipation phase will be chosen to smoothly enter the boom phase, so $C^A(T_1) = (r - \lambda_1)S^A(T_1) + PO(1 - e^{-\lambda_2(T_2 - T_1)}) + C^*$. Together these give the parameter values,

$$\begin{aligned} c_1^C &= S(T_2) \quad &; \quad c_2^C &= 0 \\ c_1^B &= S(T_1) - c_2^B \quad &; \quad c_2^B &= PO(\lambda_2 - \lambda_1)^{-1} e^{-\lambda_2 (T_2 - T_1)} \\ c_1^A &= S(0) - c_2^A \quad &; \quad c_2^A &= PO(\lambda_2 - \lambda_1)^{-1} (e^{-\lambda_2 T_2} - e^{-\lambda_2 T_1}) \end{aligned} \tag{4.5}$$

The permanent income hypothesis therefore does not hold in a capital-scarce country. Instead, consumption will change over time and depend on the level of total assets, seen directly from 4.1. Before oil is discovered consumption will begin far below its permanent level, C^*. This is because the planner has an incentive to repay initial debts, F_0, and reduce the rate of interest they face, $r(F(t))$. This brings us to our third principle,

Principle 3. ***"Consume, invest and deleverage if capital is scarce"***

i) Consumption in capital-scarce countries should begin lower than in capital abundant economies, $C_L^C(0) < C_H^C(0)$, but jump further when oil is discovered, $C_L^A(0) - C_L^C(0) > C_H^A(0) - C_H^C(0)$. It should still remain lower in absolute level, $C_L^A(0) < C_H^A(0)$.

ii) Borrowing in capital-scarce countries before a boom should initially be lower than *capital abundant countries,* $\dot{F}_L^A(0) > \dot{F}_H^A(0)$*, and if they begin to borrow it will happen near the date of extraction,* $\ddot{F}_L^A(t) < 0$ *for* $t < T_1$*.*

iii) Domestic capital in capital-scarce economies should grow over time, $\dot{K}_L^C(t) > 0$*, be accelerated during an oil boom,* $\dot{K}_L^B(t) > \dot{K}_L^C(t)$*, and comprise an increasing share of the total capital stock over time,* $\frac{d}{dt}(K(t)/\bar{S}(t)) > 0$*.*

Proof. See Appendix 3. □

This principle shows that capital-scarcity changes the way the social planner should respond to an oil discovery, illustrated in Figure 4.1.

Capital-scarcity makes debt particularly costly. Therefore, before oil is discovered capital-scarce countries (L) should prioritise saving over consumption, to repay debt and invest in capital. Consumption will thus begin lower than in a similar country with ready access to capital (H), and rise steadily over time as capital is accumulated and debt repaid (see the grey lines in Figure 4.1). It is tempting to extend this intuition after oil is discovered: that the poor should save more from marginal oil revenues. However, this is not the case.

When oil is discovered, consumption will jump because the social planner is wealthier (see the blue lines in Figure 4.1). During the anticipation phase this higher level of consumption is funded by accumulating capital and repaying debt less quickly. Once oil production begins, total assets will quickly rise as capital is accumulated and debt repaid. This allows consumption to steadily grow in expectation. At the end of the windfall total assets will continue to grow (as capital scarcity still binds), but the economy will have accumulated more assets than in a case without oil.

Principle three states that the jump in consumption after oil is discovered will be larger in a capital-scarce country than a capital abundant one. This happens for two reasons. First, the marginal utility of consumption is higher when capital is scarce, because the initial level of consumption is lower. Second, the social planner in a capital-scarce country will be significantly richer in the future because oil revenues will be used to repay debt, reduce the cost of borrowing and in turn accumulate capital, which together represent an increase in real resources available. A larger initial jump in consumption allows the planner to share these benefits over time. The initial jump in consumption is dictated by the debt elasticity, ω, rather than the level of debt itself, $\partial(C_j^A(0) - C_j^C(0))/\partial F(0) = 0$ as shown in Appendix 3.

The second part of principle three states that the social planner should borrow less before a windfall when capital is scarce than when it is abundant. This happens because borrowing is more costly, and the social planner can also finance consumption by accumulating capital less quickly (or letting it depreciate). This keeps the marginal product of capital in line with the cost of borrowing, in equation 2.9. It is interesting that there is any borrowing before a windfall at all. If the debt burden, ω, is sufficiently large - or the windfall small - then the capital-scarce country will not borrow. Consumption should begin so low that the entire jump can be financed directly from permanent income, Y. The need to borrow should also be highest just before production begins, as consumption should be steadily increasing in capital-scarce countries, $\dot{C}_L^i(t) > 0$.

The third part of this principle states that the social planner should accelerate investment during an oil boom. Part of an oil boom should be saved in foreign assets to relax the debt premium on interest rates, $r(F) = r - \omega F$. This means that capital should also grow. At every point in time, the planner should direct their savings to where it will earn the highest return: foreign assets or domestic capital. The marginal product of capital will therefore equal the cost of borrowing, 2.9. Domestic capital will also increase its share in total assets, as debt is repaid and capital accumulates.

The phase diagram in Figure 4.2 illustrates the effects of an oil discovery when capital is scarce. When the country is a net borrower, $S(t) < 0$, the economy will be governed by equations 4.1 and 4.2. These are summarised by the two blue steady state lines, $\dot{C}_D = 0$ and $\dot{F}^{A,C} = 0$ when prices are deterministic (D). These intersect at the point where all foreign debt is repaid, so $F^i = 0$ and $\bar{S}^i = K^i = S^* = (\frac{\alpha}{r+\delta})$ from Appendix A. In the absence of any shocks the social planner will choose to be on the stable (deterministic) saddle path, DD^1, which ensures the economy will move towards the steady state. This

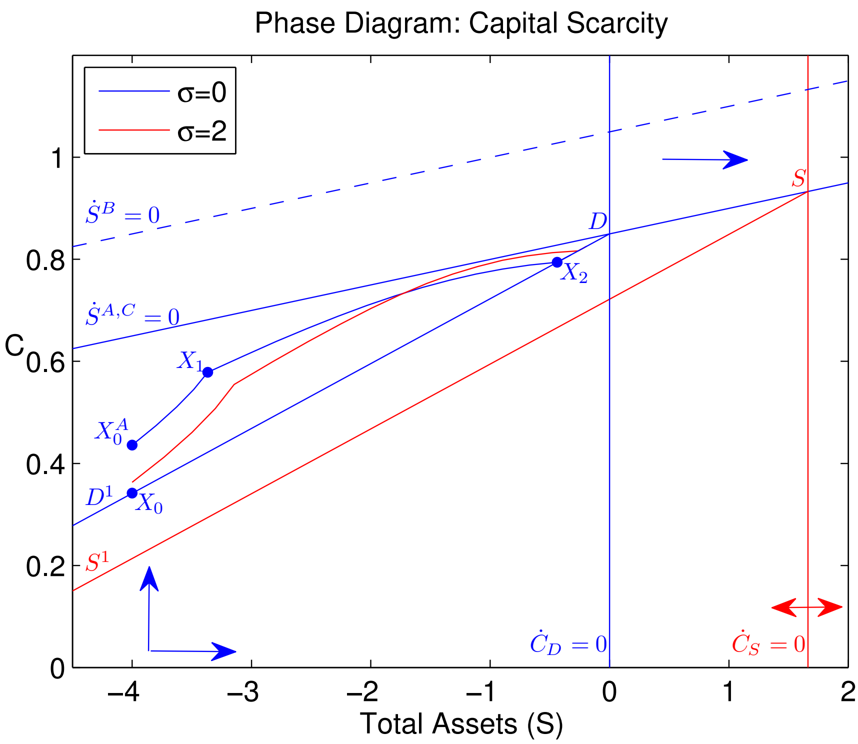

Figure 4.2: An anticipated windfall in a capital-scarce economy. Before and after the windfall the movement of consumption and assets is dictated by the solid blue lines and arrows. During the windfall the line $\dot{S}^{A,C} = 0$ shifts up to $\dot{S}^B = 0$. If oil prices are certain then the dynamics are dictated by the blue arrows and the economy follows the path $[X_0, X_0^A, X_1, X_2]$. If prices are volatile then the steady state line $\dot{C}_S = 0$ will move to the right before-, and return to $\dot{C}_D = 0$ during the windfall. This changes the path of consumption and assets to the dynamics illustrated in red.

path is a line with slope $(r - \lambda_1)$, and converges to the line $\dot{S}^{A,C} = 0$ as $\omega \to 0$. When the country is a net saver, $S(t) > 0$, the economy behaves as in Figure 3.3 and anywhere on the line $\dot{S}^{A,C} = 0$ above zero is a steady state.

Before oil is discovered the economy will begin on the stable saddle path, X_0. When oil is discovered, consumption will immediately jump and then continue to grow, X_0^A. If the windfall is small, then consumption will remain below $\dot{S}^{A,C} = 0$ and the economy will continue to accumulate assets by repaying foreign debt and investing in domestic capital, F and K. Alternatively, if the windfall is large or the debt burden small, then consumption may jump above the line $\dot{S}^{A,C} = 0$ and the planner will deplete assets: both foreign and domestic.

When the Boom begins the planner will save, X_1. The line $\dot{S}^{A,C} = 0$ will shift up, to $\dot{S}^B = 0$. The dynamics of the economy will now be dictated by a new steady state.

Consumption will be below this, so total assets will grow: repaying debt and investing in capital. If the windfall is small, then the planner will have chosen consumption to arrive on the line DD^1 at the end of the Boom, X_2. If the windfall is large, then the planner will repay all the debt and begin to accumulate assets in a sovereign wealth fund, ending the boom somewhere on $\dot{S}^{A,C} = 0$ above zero. This is discussed in the next section.

After the boom, consumption will eventually exceed that in a capital abundant economy for the same initial level of debt. In a capital abundant economy there is no incentive to repay debt. In a capital-scarce economy, the incentive to reduce the debt premium will mean that eventually the economy will converge to the steady state, D.

4.1.2 Large oil discoveries

If the discovery is large then all debt will be repaid before the windfall is exhausted.[27] At this point the economy will no longer be capital-constrained and will behave as in section 3.1. Consumption will stop growing and assets will accumulate. The economy will end the boom with a positive Future Generations Fund, on the steady state line $\dot{F}^{A,C} = 0$ above zero in Figure 4.2. The planner's behaviour will be a combination of the analysis in section 4.1.1 and section 3.1.

4.2 The Volatility Fund if capital is scarce

4.2.1 Small oil discoveries

Once again, I start with a small oil windfall so that capital will still be scarce when it is exhausted. If the windfall is volatile, then consumption and assets will evolve according to the two equations 2.10 and 2.11, reproduced below with $r = \rho$,

$$\dot{C}^i(t) = r - \rho - \tfrac{2\omega}{a} f S^i(t) + \tfrac{1}{2} a P^2 C_P^i(t)^2 \sigma^2 \tag{4.6}$$

$$\dot{S}^i(t) = r S^i(t) + PO^i - C^i(t) + C^* \tag{4.7}$$

[27] Formally, this will happen when $F(0)(\lambda_2 - \lambda_1) + PO(e^{-\lambda_1 T_2} - e^{-\lambda_1 T_1} + e^{-\lambda_2 T_1} - e^{-\lambda_2 T_2}) > 0$, which is satisfied for larger, longer windfalls and lower initial levels of debt.

It will be useful to define the instantaneous "steady state" of the system. At any point in time this state will govern the behaviour of consumption and total assets, analogous to the steady state in section 4.1. However, it will move over time as the remaining exposure to volatility changes, through $C^i_P(t)$. It is given by the two equations,

$$S^i_S(t) = \tfrac{1}{4\omega f} a^2 P^2 C^i_P(t)^2 \sigma^2 \tag{4.8}$$

$$C^i_S(t) = r S^i_S(t) + PO^i + C^* \tag{4.9}$$

The dynamics of consumption and assets around the steady state are found by solving equations 4.6 and 4.7 to give a single differential equation, $\ddot{S}^i(t) - r\dot{S}^i(t) - \frac{2\omega f}{a} S^i(t) = \frac{1}{2} a P^2 C^i_P(t)^2 \sigma^2$. As this is a non-homogeneous second-order linear differential equation, it must be solved relative to a reference point. The solution is of the form:

$$S^i(t) = c^i_1 e^{\lambda_1 t} + c^i_2 e^{\lambda_2 t} + S^i_V(t) \tag{4.10}$$

$$C^i(t) = c^i_1 (r - \lambda_1) e^{\lambda_1 t} + c^i_2 (r - \lambda_2) e^{\lambda_2 t} + C^i_V(t) \tag{4.11}$$

where $C^i_V(t) = r S^i_V(t) - \dot{S}^i_V(t)$. I choose the coefficients, c^i_1, c^i_2, to match those in equation 4.5, following the approach in section 3.2. This allows total assets to be separated into those accumulated for Future Generations, and those to manage Volatility, $S^i(t) = S^i_{FG}(t) + S^i_V(t)$. Some assets would be accumulated without an oil windfall, to overcome capital scarcity. These are included in the Future Generations assets. The Volatility assets then capture the adjustment to manage oil price volatility. Both sets of assets should be allocated between a foreign fund and domestic capital, $S^i_{FG}(t) = F^i_{FG}(t) + K^i_{FG}(t)$ and $S^i_V(t) = F^i_V(t) + K^i_V(t)$. This ensures that the asset allocation condition in equation 2.9 is satisfied, and gives rise to the next principle,

Principle 4. *"Invest part of the Volatility Fund domestically, then leave it alone"*

i) Capital-scarce countries should accumulate less in an offshore Volatility Fund than capital abundant economies, $F^i_{V,L} < F^i_{V,H}$, despite the fund accelerating the expected rate *of development.*

ii) Capital-scarce countries should also direct some precautionary savings from an oil boom to domestic investment, $K_V^i(t) > 0$.

Proof. See Appendix 4. □

As noted in Section 4.1.1, consumption should jump when oil is discovered in a capital-scarce country because the nation is wealthier. However, if oil price volatility is taken into account, then consumption should jump by less to allow for precautionary savings (see the red lines in Figure 4.1). These additional savings can be considered as a "Volatility Fund". During the anticipation phase this Fund will reduce the slowdown in capital accumulation and debt repayment that finance higher consumption (compare the red and blue lines in Figure 4.1). Once oil starts production, the additional precautionary savings in the Volatility Fund will be allocated to both accumulating capital and repaying debt, to equalize the marginal benefits of both. At the end of the windfall the additional capital and lower debt will finance higher consumption.

Principle four starts by comparing the size of Volatility Funds in capital-abundant and capital-scarce countries. Capital-scarcity reduces the precautionary motive, and thus the size of Volatility Fund, because it increases the marginal utility from consuming more.[28] Put simply, poor countries benefit more from extra consumption. Capital-scarce countries should therefore have a large Future Generations Fund, and a relatively small Volatility Fund.

Principle four also states that part of the Volatility Fund should be invested domestically if capital is scarce. Holding all of the Volatility assets in an offshore sovereign wealth fund, $F_V > 0$ and $K_V = 0$, will reduce the cost of capital, $r(F) = r - \omega(F_{FG} + F_V)$. More domestic investment will be profitable at this lower cost of capital. Thus, some of the Volatility Funds should be invested domestically to keep the marginal products of capital and foreign assets equal, from equation 2.9. While not captured in the model, in practice

[28]Note that this analysis assumes prudence is constant, regardless of consumption. This makes the effects of volatility more clear. If prudence is higher at low levels of consumption (such as if relative risk aversion is constant), then higher prudence may outweigh the costs of lower consumption.

domestic fixed capital is less liquid than foreign assets. This should not be a major concern as the Volatility Fund should be treated as a source of permanent income, rather than a "shock absorber", as argued in principle two.

The phase diagram in Figure 4.2 also displays the effects of oil price volatility. The stochastic case (red, subscript S) illustrates the dynamics in equations 4.10 and 4.11. It differs from the deterministic case in section 4.1 in an important way: volatile oil prices cause the vertical "steady state" line (labeled $\dot{C}_S = 0$, and given in equation 4.8) to move to the right, reflecting the higher steady state level of assets needed to compensate for volatility.

When oil is discovered the social planner becomes exposed to oil price volatility. The line $\dot{C}_S = 0$ will jump to the right, as will the stable saddle path, SS^1, which encourages less consumption relative to the deterministic case for any given level of assets. Both lines will continue to move right until oil production begins, when the exposure to oil price volatility, $C_P^i(t)$, is highest (see Figure 3.1).

During the Boom phase the line $\dot{S}^{A,C} = 0$ will shift up to $\dot{S}^B = 0$, reflecting the inflow of oil revenues. At the same time both the $\dot{C}_S = 0$ line and the saddle path, SS^1, will move left as the subsoil exposure to oil price volatility falls. Consumption will remain lower than the deterministic case for any given level of assets throughout this phase. After all oil is extracted there will no longer be any exposure to oil price risk and the $\dot{C}_S = 0$ line will converge to $\dot{C}_D = 0$ at $t = T_2$. At this point the economy will meet the line DD^1 at a higher level of both consumption and assets than the deterministic case, due to lower consumption throughout the boom.

4.2.2 Large oil discoveries

If the discovery is large and volatile, then all debt should be repaid and the desired capital stock reached, before accumulating a Future Generations Fund. A large discovery will start in capital scarcity, and finish in capital abundance. At the start of the windfall the economy will behave as in section 4.2.1. At some point during the windfall total assets will accumulate far enough that debt is repaid and capital-scarcity is relaxed, $F^B(t^*) = 0$ at some point $T_1 < t^* < T_2$. For the rest of the windfall the economy will behave as if it were capital-abundant, from section 3.2. The vertical steady state line, $\dot{C}_S = 0$, will disappear as capital is no longer scarce, $\omega = 0$ in equation 4.8. Thus, the social planner never actually reaches this "target". Consumption will be chosen to grow smoothly throughout the windfall and finish on the steady state line at the end of the boom, $\dot{S}^{A,C} = 0$ at $t = T_2$ in equation 4.9.

5 Conclusion

This paper considers whether policymakers should spend, save or invest a volatile oil windfall. It extends existing literature by showing that the principal in Volatility Funds should not be depleted when oil prices fall, because policymakers don't know when, or if, the price will rise again. Consumption should adjust instead. A Volatility Fund should be built in advance if the windfall is anticipated, and treated as a source of permanent income, rather than a temporary buffer for smoothing out oil shocks. This approach could be consistent with investing the fund in income-generating assets with a long horizon, though investment horizon is beyond the scope of this current work. To establish these findings the paper develops a framework that nests a variety of existing results, which I present in four principles. The first two apply to developed countries: i) smooth consumption using a Future Generations Fund, and ii) build a Volatility Fund quickly, then leave it alone. The second two apply to developing, capital-scarce countries: iii) consume, invest and deleverage if capital is scarce, and i) invest part of the Volatility Fund domestically, then leave it alone. These principles also suggest a number of extensions. Adding nominal rigidities would introduce a role for monetary policy, and possibly soften the prohibition on depleting Volatility Fund principal if monetary policy is constrained. Adding physical adjustment costs to public spending might similarly partially offset the prohibition. Another extension could introduce political economy. Accumulating Volatility or Future Generations funds may be less advisable if there is a probability that they will be raided by politicians in the future. A formal treatment of this, balancing relatively easily raided sovereign wealth fund assets against illiquid domestic investments, could be the subject of future work.

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Appendix A: Consumption and Total Assets

This appendix derives the equations governing the expected dynamics of consumption and total assets. I begin by taking the first-order conditions of the Hamilton-Jacobi-Bellman equation in 2.6 and 2.7 with respect to consumption, investment, foreign assets, and domestic capital, where A.3 and A.4 make use of the envelope condition,

$$U'(C)e^{-\rho t} - J_F = 0 \tag{A.1}$$

$$-J_F + J_K = 0 \tag{A.2}$$

$$\begin{aligned} 0 + J_{tF} + J_{FF}(r(F)F + PO + Y(K) - C - I) + J_F(r + r_F F) \\ + J_{KF}(I - \delta K) + \tfrac{1}{2} J_{PPF}\sigma^2 P^2 &= 0 \end{aligned} \tag{A.3}$$

$$\begin{aligned} J_{tK} + J_{FK}(r(F)F + PO + Y(K) - C - I) + Y_K J_F \\ + J_{KK}(I - \delta K) - \delta J_K + \tfrac{1}{2} J_{PPK}\sigma^2 P^2 &= 0 \end{aligned} \tag{A.4}$$

The results in equations 2.8 and 2.9 come from combining equations A.2, A.3 and A.4 with $dJ_s = J_{sF}dF + J_{sK}dK + J_{sP}dP + J_{st}dt + \frac{1}{2}J_{sPP}dP^2$ for $s = [F, K]$.

Substituting A.1 and 2.7 into 2.6, and using $U(C) = \frac{1}{-a}e^{-aC}$ gives the following optimality condition, which the value function must satisfy,

$$-\frac{1}{a}J_F + \{J_t + J_F(r(F)F + PO + Y(K) - C - I) + J_K(I - \delta K) + \frac{1}{2}J_{PP}\sigma^2 P^2\} = 0 \quad \text{(A.5)}$$

The social planner's problem can be summarised in terms of the overall evolution of total assets, $\bar{S} = F + K$, as foreign assets and capital are linked by equation 2.9. Substituting 2.3 into 2.2 gives the evolution of total assets (equation A.6 below). Total assets will be constant at the steady state, S^*, when capital scarcity is overcome and oil income is exhausted, $F^* = 0$ and $O^* = 0$. I define technology so that output is standardised at this point, $A = (\frac{r+\delta}{\alpha})^\alpha$ so $Y^* = 1$ and $K^* = (\frac{\alpha}{r+\delta})$.

$$d\bar{S}^i(t) = (r(F^i(t))F^i(t) + P(t)O^i + Y(K^i(t)) - C^i(t) - \delta K(t))dt \quad \text{(A.6)}$$

To analyse the dynamics of consumption and total assets I need to linearise this equation around S^* to give the following, where $S = \bar{S} - S^*$.

$$dS^i(t) = \Big(rF^i(t) + P(t)O^i + Y'(K^*)(K^i(t) - K^*) - (C^i(t) - C^*) - \delta(K^i(t) - K^*)\Big)\, dt \quad \text{(A.7)}$$

Both foreign assets and domestic capital can be expressed as a share of total assets. At any point in time the optimal level of domestic capital can be linked to the level of foreign assets, $K(F) = (\frac{r+\delta-\omega F}{\alpha})^{1/(\alpha-1)}A^{-1/(\alpha-1)}$ from equation 2.9. Therefore, I can express total assets as a non-linear function of foreign assets, $S = S(F)$. However, I need an expression for foreign assets as a function of total assets, $F = F(S)$. The only way to proceed with a tractable solution is to linearise $S(F)$, which I do around the deterministic steady state S^*. Linearisation yields $S = f^{-1}(F - F^*)$ where $f^{-1} = (1 + \frac{\omega}{\alpha(1-\alpha)}(\frac{\alpha}{r+\delta})^2)$. Using this, $\bar{S} = F + K$ and A.7 gives the relationship,

$$\frac{1}{dt}dS^i(t) = \theta S^i(t) + P(t)O^i - C^i(t) + C^* \quad \text{(A.8)}$$

where $\theta = rf + (Y'(K^*) - \delta)(1 - f)$ is the overall rate of return on total assets, and $C^* = Y^* - \delta K^*$ is the steady state level of consumption. Finally, making use of equation 2.9 at the steady state gives $r = Y'(K^*) - \delta$, so $\theta = r$ and I have the budget constraint in equation 2.11. This simple expression is possible because I have linearised the relationship between foreign assets and domestic capital in equation 2.9 to give, $F = f/(1-f)(K-K^*)$.

The Euler equation in 2.10 is found by taking the expected rate of change with respect to time of A.1, given in A.9 below, and combining it with expressions 2.8, $U(C) = \frac{1}{-a}e^{-aC}$, and $S = f^{-1}(F - F^*)$.

$$\frac{E[dJ_F]/dt}{J_F} = \frac{U''(C)}{U'(C)}E[dC]/dt - \rho + \tfrac{1}{2}\frac{U'''(C)}{U'(C)}C_P^2\sigma^2P^2dt \tag{A.9}$$

Appendix B: The Permanent Income Hypothesis

This appendix derives an explicit expression for the value function, $J(F, K, P, t)$ in equation 2.1. This is done in the special case when preferences are CARA, $U(C) = \frac{1}{-a}\exp(-aC)$, oil prices are certain, $\sigma = 0$, and the interest rate is constant, $\omega = 0$. The value function must satisfy the optimality condition in equation A.5, which reduces to,

$$-\tfrac{1}{a}J_F + \{J_t dt + J_F(rF + PO + Y - C)\} = 0 \tag{A.10}$$

Let us define total wealth at time t, $W(t)$, to be the sum of foreign assets, $F(t)$, the present value of oil income $V^i(t)$, and the present value of permanent non-oil income less depreciation $C^*/r = (Y^* - \delta K^*)/r$. The present value of oil income is found by discounting at the risk-free rate, and will depend on whether the economy is in the Anticipation, Boom or Constant income stage of the oil windfall,

$$\begin{aligned} V^C(t) &= 0 && \text{for } T_2 \le t \\ V^B(t) &= PO(1 - e^{-r(T_2-t)})/r && \text{for } T_1 \le t < T_2 \\ V^A(t) &= PO(e^{-r(T_1-t)} - e^{-r(T_2-t)})/r && \text{for } 0 \le t < T_1 \end{aligned} \tag{A.11}$$

The explicit form of the value function will be as follows, which can be verified by substitution into the partial differential equation A.10,

$$\begin{aligned} J(F,P,t) &= \frac{-A}{a} e^{-ra(W(t))-\rho t} \qquad \text{(A.12)} \\ \text{where} \quad A &= \tfrac{1}{r}\exp((r-\rho)/r) \end{aligned}$$

Consumption will satisfy the permanent income hypothesis. From equation A.1, consumption will be a fixed proportion of total wealth, $C(t) = rW(t)$ when $r = \rho$. This means that as oil wealth $V(t)$ is extracted, it is converted into foreign assets $F(t)$, and only the permanent income is consumed. The partial derivatives $\partial C^i(t)/\partial P(t)$ are the same as those in equation 3.6.

Appendix C: Calibration

Parameter	Value	Parameter	Value
a	1.00	f	0.49
ρ	0.05	$F_j(0)$	-1.96
$Y(K^*)$	1.00	P_0	1.00
α	0.40	O	0.2
δ	0.03	T_1	4
ω	0.01	T_2	16
$S_L(0)$	-4.00	σ	Various

Table A.1: Values used in the calibrated simulations.

Appendix 1: Proof of Principle 1

i) Follows directly from equation 3.1.

ii) The present value of oil income is given in equation A.11. Combining this with equations 3.3 and 3.5 gives the result.

iii) From $F^i(t) = a_1^i e^{rt^i} + a_2^i$ and equation 3.5, assets will evolve according to,

$$\begin{aligned} F^C(t) &= F(T_2) && \text{for} \quad T_2 \leq t \\ F^B(t) &= F(T_1) + PO(e^{rt} - e^{rT_1})e^{-rT_2}/r && \text{for} \quad T_1 \leq t < T_2 \\ F^A(t) &= F(0) + PO(e^{rt} - 1)(e^{-rT_2} - e^{-rT_1})/r && \text{for} \quad 0 \leq t < T_1 \end{aligned} \tag{A.13}$$

Therefore, $\dot{F}^A(t) = POe^{rt}(e^{-rT_2} - e^{-rT_1}) < 0$, $\dot{F}^B(t) = POe^{-r(T_2-t)} > 0$ and $\dot{F}^C(t) = 0$.

iv) $\dot{F}^A(t) < 0$ is the amount borrowed in period $0 \leq t < T_1$. From the results in ii), $d\dot{F}^A(t)/dT_2 = -rPOe^{-r(T_2-t)} < 0$, so a longer windfall will cause the social planner to borrow more before the windfall. $\dot{F}^B(t) > 0$ is the amount saved in period $T_1 \leq t < T_2$. Also from part ii), $d\dot{F}^B(t)/dT_2 = -rPOe^{-r(T_2-t)} < 0$, so a longer windfall reduces the amount saved each period during the windfall.

v) When interest rates are constant and goods can be freely bought and sold at the world price, then $\omega = 0$ in equation 2.9. Capital will therefore be constant, $K = K^* = (\frac{r+\delta}{\alpha A})^{1/(\alpha-1)}$. Technology is defined to normalise $Y(K^*) = 1$ so $K^* = \frac{\alpha}{r+\delta}$.

Appendix 2: Proof of Principle 2

The particular solution to equation 3.9, so that a_1^i, a_2^i are defined in equation 3.5, is,

$$F_V^C(t) = F_V^B(T_2) \tag{A.14}$$

$$\begin{aligned} F_V^B(t) = {} & \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r^2}\Big(t^B B_1 - B_2(1 + r(T_2 - T_1 - t^B))(e^{rt^B} - 1) \\ & -B_3(e^{2rt^B} - 1)\Big) + F_V^A(T_1) \end{aligned} \tag{A.15}$$

$$F_V^A(t) = \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r^2}\Big(A_1(e^{rt} - 1) - A_2(e^{2rt} - 1)\Big) \tag{A.16}$$

where $t^B = t - T_1$, $B_1 = r(1 + 2e^{-r(T_2-T_1)})$, $B_2 = 2e^{-r(T_2-T_1)}$, $B_3 = \frac{1}{2}e^{-2r(T_2-T_1)}$ $A_1 = 2\Big((e^{-rT_1} - e^{-rT_2}) - (T_2 - T_1)re^{-rT_2}\Big)$, $A_2 = \frac{1}{2}(e^{-rT_1} - e^{-rT_2})^2$, and all $A_j, B_j \geq 0$. This can be verified by substitution into $\ddot{F}^i(t) - r\dot{F}^i(t) = \frac{1}{2}aP^2\sigma^2 C_P^i(t)^2$.

The proof of this principle relies on the following two Lemmas, after which I turn to

Principle 2:

Lemma 1. *The balance of the Volatility Fund never decreases in expectation,* $\dot{F}^i_V(t) > 0$ *for all* $t > 0$.

Proof. The time derivative of the fund during each phase is,

$$\dot{F}^C_V(t) = 0 \tag{A.17}$$

$$\dot{F}^B_V(t) = \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r^2}\Big(B_1 - rB_2(1 + (T_2 - T_1 - t^B)re^{rt^B}) - B_3 2re^{2rt^B}\Big) \tag{A.18}$$

$$\dot{F}^A_V(t) = \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r^2}\Big(A_1 re^{rt} - A_2 2re^{2rt}\Big) \tag{A.19}$$

During the Anticipation phase I begin by showing that $\dot{F}^A_V(T_1) > 0$,

$$\begin{aligned}\dot{F}^A_V(T_1) &= \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r^2}\Big(A_1 re^{rT_1} - A_2 2re^{2rT_1}\Big)\\ &= \tfrac{1}{2}aP^2O^2\sigma^2\tfrac{1}{r}\Big(1 - (2r(T_2 - T_1) + e^{-r(T_2-T_1)})e^{-r(T_2-T_1)}\Big)\end{aligned}$$

Now, when $T_2 = T_1$ then $\dot{F}^A_V(T_1) = 0$. Also, by l'Hopital's rule, $\lim_{(T_2-T_1)\to\infty}\dot{F}^A_V(T_1) = \frac{1}{2}aP^2O^2\sigma^2\frac{1}{r} > 0$. As $\ddot{F}^A_V(T_1) < 0$ for all T_1 then $\dot{F}^A_V(T_1) > 0$. It is also easy to see that $\dot{F}^A_V(t) > \dot{F}^A_V(T_1)$ for all $t < T_1$. Thus, $\dot{F}^A_V(t) > 0$.

During the Boom phase it is straightforward to show that $\dot{F}^B_V(T_2) > 0$ by substitution (that is when $t^B = T_2 - T_1$). Furthermore, $\dot{F}^B_V(t) > \dot{F}^B_V(T_2)$ for all $t < T_2$. Thus, $\dot{F}^B_V(t) > 0$. □

Lemma 2. *The rate of increase of the Volatility Fund slows over time,* $\ddot{F}^A_V(t), \ddot{F}^B_V(t) < 0$ *for* $t < T_2$.

Proof. The second derivative of $F^i_V(t)$ with respect to time is,

$$\ddot{F}^C_V(t) = 0 \tag{A.20}$$

$$\ddot{F}^B_V(t) = \tfrac{1}{2}aP^2O^2\sigma^2\Big(B_2(1 - r(T_2 - T_1 - t^B))e^{rt^B} - B_3 4e^{2rt^B}\Big) \tag{A.21}$$

$$\ddot{F}^A_V(t) = \tfrac{1}{2}aP^2O^2\sigma^2\Big(A_1 e^{rt} - A_2 4e^{2rt}\Big) \tag{A.22}$$

During the Anticipation phase I begin by showing that $\ddot{F}_V^A(0) < 0$. When $t = 0$,

$$\ddot{F}_V^A(0) = aP^2O^2\sigma^2\Big((e^{-rT_1} - e^{-rT_2}) - (T_2 - T_1)re^{-rT_2} - (e^{-rT_1} - e^{-rT_2})^2\Big)$$

If $T_2 = T_1$then $\ddot{F}_V^A(0) = 0$. So, if $T_2 > T_1$ then $\ddot{F}_V^A(0) < 0$. Now, I also know from inspection of equation A.22 that $\ddot{F}_V^A(t) < \ddot{F}_V^A(0)$ for all $0 < t < T_1$. Therefore $\ddot{F}_V^A(t) < 0$ for all $0 < t < T_1$.

During the Boom phase I begin by showing that $\ddot{F}_V^B(T_1) < 0$ for $r < \frac{2}{3}$ (i.e. $t^B = 0$),

$$\ddot{F}_V^B(T_1) \quad = \quad \tfrac{1}{2}aP^2O^2\sigma^2\Big(r(1 + 2e^{-r(T_2-T_1)})(1 - r(T_2 - T_1)) - 2e^{-2r(T_2-T_1)}\Big)$$

If $T_2 = T_1$ then $\ddot{F}_V^B(T_1) < 0$ if $r < \frac{2}{3}$. If $T_2 > T_1$ then $\ddot{F}_V^B(T_1)$ will become more negative. I can also see from inspection of equation A.21 that $\ddot{F}_V^B(t) < \ddot{F}_V^B(T_1)$ for all $T_1 < t < T_2$. Therefore $\ddot{F}_V^B(t) < 0$ for all $T_1 < t < T_2$. □

Build a Volatility Fund quickly:

i) The Volatility Fund begins at $F_V^A(0) = 0$, as prior to discovering oil there is no exposure to volatility. The expected balance of the fund is also always increasing, $\dot{F}_V^i(t) > 0$ for all $0 < t < T_2$ by Lemma 1. Therefore $F_V^i(t) > 0$ for all $t > 0$.

ii) The balance of the Volatility Fund increases for all $0 < t < T_2$, $\dot{F}_V^A(t), \dot{F}_V^B(t) > 0$ by Lemma 1. However, the rate it increases will diminish over time, $\ddot{F}_V^A(t), \ddot{F}_V^B(t) < 0$ for $t < T_2$ by Lemma 2. Thus, it receives relatively more savings early in the windfall.

Then leave it alone:

iii) I know that $F_V^i(0) = 0$. Thus, a contemporaneous oil price shock will not affect the stock of assets in the Volatility Fund, only the subsequent path of consumption.

iv) The marginal effect of a change in oil prices on consumption falls as oil is extracted, $C_P^B(t) = O(1 - e^{-r(T_2-t)}) \to 0$ as $t \to T_2$ from equation 3.6. The result then follows directly from equation 3.7, capturing a diminishing incentive for precautionary savings.

v) This follows from Lemma 1.

Appendix 3: Proof of Principle 3

i) Phase C describes the economy in the absence of oil. If capital is scarce (L), consumption and assets will lie on the stable saddle path during this phase, $C_L^C(t) = (r - \lambda_1)S_L^C(t) + C^*$, from equations 4.3, 4.4 and 4.5. If capital is abundant (H) they will lie on the steady state line, $C_H^C(t) = rS_H^C(t) + C^*$, from equation 3.2 with $\dot{F} = O = 0$. As $\lambda_1 < 0$ and $S_L^C(0) < S_H^C(0) < 0$ (as the capital stock will also be lower when capital is scarce), $C_L^C(0) < C_H^C(0)$ for any given level of C^*.

Next I note that $C_j^C(0)$ is the level of consumption given the starting level of assets, $S_j(0)$, during the Constant income phase for $j = [L, H]$. This is the level of consumption that would hold at time $t = 0$ if there is no windfall. Now, I have,

$$\begin{aligned}
C_L^A(0) - C_L^C(0) &= (r - \lambda_1)S_L(0) + PO(e^{-\lambda_2 T_1} - e^{-\lambda_2 T_2}) + C^* \\
&\quad -S_L(0)(r - \lambda_1) - C^* \\
&= PO(e^{-\lambda_2 T_1} - e^{-\lambda_2 T_2}) \\
C_H^A(0) - C_H^C(0) &= rS_H(0) + PO(e^{-rT_1} - e^{-rT_2}) + C^* - rS_H(0) - C^* \\
&= PO(e^{-rT_1} - e^{-rT_2})
\end{aligned}$$

Now, $(e^{-rT_1} - e^{-rT_2}) - (e^{-\lambda_2 T_1} - e^{-\lambda_2 T_2}) < 0$, for $\omega > 0$ as $\lambda_2 = \frac{1}{2}r \pm \frac{1}{2}\sqrt{r^2 + 8\omega f/a}$. So on discovering oil consumption jumps more in capital-scarce than capital abundant economies.

Finally, let the difference between consumption in capital abundant and capital-scarce economies be, $c^i(t) = C_H^i(t) - C_L^i(t)$. The associated difference in assets is $s^i(t) = S_H^i(t) - S_L^i(t)$. The difference in consumption will satisfy,

$$\begin{aligned}
\dot{c}^i(t) &= \dot{C}_H^i(t) - \dot{C}_L^i(t) \\
&= \tfrac{2\omega}{a} f S_L^i(t) \\
&= -\tfrac{2\omega}{a} f s^i(t) + \tfrac{2\omega}{a} f S_H^i(t) < 0
\end{aligned}$$

So, consumption in the capital-scarce economy grows more quickly than consumption in the capital abundant economy. The difference in assets will satisfy $\dot{s}^i(t) = rs^i(t) - c^i(t)$. These dynamics are summarised by the differential equation $\ddot{s}^i(t) - r\dot{s}^i(t) - \frac{2\omega}{a}s^i(t) = -\frac{2\omega}{a}fS^i_H(t)$. The explicit solution to this is,

$$\begin{aligned} s^i(t) &= k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} + a_1 e^{rt} + a_2 && \text{(A.23)} \\ &= k_1 e^{\lambda_1 t} + k_2 e^{\lambda_2 t} + S^A_H(t) \\ c^i(t) &= (r-\lambda_1)k_1 e^{\lambda_1 t} + (r-\lambda_2)k_2 e^{\lambda_2 t} + ra_2 && \text{(A.24)} \end{aligned}$$

To find the coefficients k_1, k_2 I impose two constraints. First, at $t = 0$, $0 = k_1 + k_2 + S_L(0)$ and $s(0) > 0$ as the initial domestic capital stock will be lower when capital is scarce. Second, the difference between the capital-scarce and capital abundant economies will disappear if foreign assets ever reach zero. So, if for any point in time $t = t^* > 0$, $S^i_H(t^*) = S^i_L(t^*) = s^i(t^*) = 0$, then the difference in consumption will also be zero, $c^i(t^*) = 0$. This gives $k_1 = -S_L(0) - k_2$ and $k_2 = ra_2(\lambda_2 - \lambda_1)^{-1}e^{-\lambda_2 t^*}$. Substituting this into equation A.24 gives $c^A(0) = \lambda_1 S_L(0) + PO(e^{-rT_1} - e^{-rT_2})(1 - e^{-\lambda_2 t^*}) > 0$. Thus, the absolute level of consumption on discovering oil in a capital-scarce economy will be lower than in a capital abundant economy, $C^A_H(0) > C^A_L(0)$.

ii) From part i) I have $C^A_H(0) > C^A_L(0)$. As $\dot{S}^A_j(t) = rS^A_j(t) + C^* - C^A_j(t)$ for $j = [L, H]$, so $\dot{S}^A_H(0) < \dot{S}^A_L(0)$. I also know that $F^i_H(t) = S^i_H(t)$ and $F^i_L(t) = fS^i_L(t)$ where $f < 1$. So, $\dot{F}^A_L(0) > \dot{F}^A_H(0)$ and capital scarce countries will initially borrow less.

As the Anticipation phase proceeds, the amount saved by a capital-scarce economy will decrease (or the rate of borrowing will increase). From equation 4.3 I have $S^i(t^i) = c^i_1 e^{\lambda_1 t^i} + c^i_2 e^{\lambda_2 t^i}$ and $\ddot{S}^A(t) = c^A_1 \lambda_1^2 e^{\lambda_1 t} + c^A_2 \lambda_2^2 e^{\lambda_2 t}$. I know that $c^A_1 = S(0) - c^A_2$ and $c^A_2 = -PO(e^{-\lambda_2 T_1} - e^{-\lambda_2 T_2})(\lambda_2 - \lambda_1)^{-1} < 0$ from equation 4.5. If a windfall is small, its present value will be much less than is needed to overcome capital scarcity, $PO(e^{-rT_1} - e^{-rT_2})/r \ll |S(0)|$. This is because the country will also be saving from income Y and I assume that $S(T_2) < 0$. As $c^A_2 \approx -PO(e^{-rT_1} - e^{-rT_2})/r$ and $S(0) < 0$ I have, $c^A_1 = S(0) - c^A_2 < 0$ for a small windfall. So, $\ddot{S}^A(t) = c^A_1 \lambda_1^2 e^{\lambda_1 t} + c^A_2 \lambda_2^2 e^{\lambda_2 t} < 0$ and in turn $\ddot{F}^A(t) < 0$.

iii) In equation 4.3 I see that $\dot{S}^C(t) > 0$ for all t. I also know that $S^C(t) = (1 - f)(K^C(t) - K^*)$. As $f < 1$ I have $\dot{K}_L^C(t) > 0$. Next, from equation 4.3 I have $\dot{S}^i(t) = c_1^i \lambda_1 e^{\lambda_1 t^i} + c_2^i \lambda_2 e^{\lambda_2 t^i}$. Considering how total assets evolve with or without an oil boom, starting at the same initial level of assets S_0, from equation 4.5 I have $c_1^C = S_0$, $c_2^C = 0$, $c_1^B = S_0 - c_2^B$ and $c_2^B > 0$. So, $\dot{S}^B(t) > \dot{S}^C(t)$ as $c_1^B < c_1^C$, $\lambda_1 < 0$, $c_2^B > c_2^C$ and $\lambda_2 > 0$. Therefore $\dot{K}^B(t) > \dot{K}^C(t)$. Finally, I know that $\bar{S}(t) = F(t) + K(t)$. I also can see that $\dot{F}(t) = \frac{\alpha(1-\alpha)}{2\omega} AK(t)^{\alpha-2}\dot{K}(t)$ from equation 2.9. At $K(t) = K^* = \frac{\alpha}{r+\delta}$ and using $A = (\frac{r+\delta}{\alpha})^\alpha$ from Appendix A, $\frac{\alpha(1-\alpha)}{2\omega} AK(t)^{\alpha-2} = \frac{(1-\alpha)}{2\omega}\frac{(r+\delta)^2}{\alpha}$. For $\alpha \approx (1 - \alpha)$ and r, δ and ω of similar order of magnitude, $\frac{\alpha(1-\alpha)}{2\omega} AK^{\alpha-2} < 1$. So, near K^* domestic capital grows more quickly than foreign assets, $\dot{F}(t) < \dot{K}(t)$. Domestic capital therefore increases its share in total assets, $\frac{d}{dt}(K(t)/\bar{S}(t)) > 0$. This will not necessarily hold if $K(t)$ is significantly below K^*, which I abstract from in our linearised analysis.

Appendix 4: Proof of Principle 4

I begin with the following expressions for $S_V^i(t)$:

$$S_V^C(t) = S_V^B(T_2) \quad \text{(A.25)}$$

$$S_V^B(t) = \frac{1}{4\omega f} a^2 P^2 O^2 \sigma^2 \Big(B_1(e^{2rt^b} - e^{\lambda_1 t^b}) - B_2(e^{rt^b} - e^{\lambda_1 t^b}) - B_3(e^{\lambda_2 t^b} - e^{\lambda_1 t^b}) + (1 - e^{\lambda_1 t^b}) \Big) \quad \text{(A.26)}$$

$$S_V^A(t) = \frac{1}{4\omega f} a^2 P^2 O^2 \sigma^2 \Big(A_1(e^{2rt} - e^{\lambda_1 t}) - A_2(e^{\lambda_2 t} - e^{\lambda_1 t}) \Big) \quad \text{(A.27)}$$

Where $t^b = t - T_1$, $B_1 = \frac{\omega f/a}{\omega f/a - r^2} e^{-2r(T_2-T_1)}$, $B_2 = 2e^{-r(T_2-T_1)}$, $B_3 = \frac{r^2}{\omega f/a - r^2}\frac{2r-\lambda_1}{\lambda_2-\lambda_1} e^{-\lambda_2(T_2-T_1)}$, $A_1 = \frac{\omega f/a}{\omega f/a - r^2}(e^{-rT_1} - e^{-rT_2})^2$ and $A_2 = \frac{e^{-\lambda_2 T_1}}{(\lambda_2-\lambda_1)}\Bigg\{ \frac{(2r-\lambda_1)}{\omega f/a - r^2} r^2 \Big(1 - 2e^{-r(T_2-T_1)} + e^{-\lambda_2(T_2-T_1)}\Big) + 2r(1 - e^{-r(T_2-T_1)}) \Bigg\}$, where all $B_j, A_k > 0$ for $\omega f/a - r^2 > 0$. Also, I have, $C_V^i(t) = rS_V^i(t) - \dot{S}_V^i(t)$. This can be verified by substitution into equations 4.6 and 4.7.

i) The Volatility Fund is the difference between the total level of foreign assets in a stochastic and a deterministic setting (the Future Generations Fund), $F_{V,j}^i = F_j^i -$

$F^i_{FG,j}$, for both capital-scarce (L) and capital-abundant (H) economies, $j = [L, H]$. This Volatility Fund will be a fraction of the total level of assets accumulated to manage volatility, $F^i_{V,j} = fS^i_{V,j}$, which satisfies the dynamic equation, $\dot{S}^i_{V,j} = rS^i_{V,j} + PO^i + C^* - C^i_{V,j}$ for both capital abundant and capital-scarce economies. The volatility effect on consumption, $C^i_{V,j} = C^i_j - C^i_{FG,j}$, will satisfy the following equations, for capital abundant and capital-scarce economies respectively,

$$\begin{aligned} \dot{C}^i_{V,H} &= \dot{C}^i_H - \dot{C}^i_{FG,H} \\ &= \tfrac{1}{2}aP^2C^i_P(t)^2\sigma^2 \end{aligned} \tag{A.28}$$

$$\begin{aligned} \dot{C}^i_{V,L} &= \dot{C}^i_L - \dot{C}^i_{FG,L} \\ &= \{-\tfrac{2\omega f}{a}S^i_L + \tfrac{1}{2}aP^2C^i_P(t)^2\sigma^2\} - \{-\tfrac{2\omega f}{a}S^i_{FG,L}\} \\ &= -\tfrac{2\omega f}{a}S^i_{V,L} + \tfrac{1}{2}aP^2C^i_P(t)^2\sigma^2 \end{aligned} \tag{A.29}$$

The rate of change of the volatility adjustment to consumption is higher in capital-abundant economies, $\dot{C}^i_{V,L}(t) < \dot{C}^i_{V,H}(t)$ for all $t > 0$. This follows from equations A.25 to A.27, which show that $S^i_{V,j}(t) > 0$ for all $t > 0$, and equation A.29. This is consistent with precautionary savings building up a positive Volatility Fund.

Therefore, the adjustment to consumption from volatility will be greater in capital-abundant economies during the early years of the windfall, $C^i_{V,H}(t) < C^i_{V,L}(t) < 0$ for t less than some t^V. The volatility effect on total assets is zero before oil is discovered, for both capital-scarce and capital-abundant economies, $S_{V,j}(0) = 0$ for $j = [L, H]$. The "budget constraint" tying together the volatility adjustments to consumption and total assets is also the same in both capital-abundant and capital-scarce economies, $\dot{S}^i_{V,j} = rS^i_{V,j} + PO^i + C^* - C^i_{V,j}$ for $j = [L, H]$. If both the initial state and the budget constraint are the same in both countries, but the consumption adjustment grows more quickly in the capital-abundant economy, then its initial level must be lower. So, $C^i_{V,H}(t) < C^i_{V,L}(t) < 0$.

As a result, capital-abundant economies will accumulate a larger volatility fund than capital-scarce economies, $F^i_{V,H} > F^i_{V,L}$. The overall amount of assets accumulated to manage volatility will grow faster in a capital-abundant country, $\dot{S}^i_{V,H} > \dot{S}^i_{V,L}$. All of these assets in a capital-abundant economy are accumulated in a Volatility Fund, $F_{V,H} = S_{V,H}$, while only a fraction are in a capital-scarce economy, $F_{V,L} = fS_{V,L}$ where $f \in [0,1]$. Thus, capital-abundant economies will accumulate a larger Volatility Fund than capital-scarce economies. Over time, the larger Volatility Fund in the capital-abundant country will generate interest payments, which will finance a larger volatility effect on consumption, $C^i_{V,H}(t) > C^i_{V,L}(t)$ for $t > t^V$.

This holds despite the Volatility Fund accelerating the pace of development in expectation. This follows directly from equation 4.6, where volatility increases the rate of change of consumption $\frac{\partial}{\partial\sigma}\dot{C}^i(t) > 0$. This means that consumption will be lower in the short-run, to finance precautionary savings. These precautionary savings lead to the accumulation of a Volatility Fund, $F^i_V(t) > 0$ from equations A.25 to A.27, in addition to the future generations fund. This additional Volatility Fund will reduce capital scarcity, and generate income that can finance a higher level of consumption in the future.

ii) From equations A.27 to A.25 I have that $S^i_V(t) > 0$ for all $t > 0$. I also know that $K^i_V(t) - K^{*i}_V = (1-f)S^i_V(t)$ where $f \in [0,1]$. Finally, I know that the volatility adjustment to the capital stock in the deterministic steady state will be zero, as I assume that there is no volatile income at this point, $K^{*i}_V = 0$. So, a volatile oil windfall should increase the stock of capital, $K^i_V(t) > 0$.